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A Survey of [AJS]

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This is a survey of the work [AJS] by H.H. Andersen, J.C. Jantzen and W. Soergel. There are also excellent expositions by the authors [A2], [S1], [S2], of which [A2] includes the entire aspect of Lusztig's program.

During the AMS Summer Institute 1986 at Arcata I had an opportunity to ask G. Lusztig how he had come to his conjectural formula [L1] that should describe the irreducible characters of simple \mathbb{F}_p -groups in terms of the Kazhdan-Lusztig polynomials. He kindly explained me the idea, that is in [H], and said it would be easier to relate the conjecture to his analogous conjecture for affine Kac-Moody Lie algebras than to derive the exact formula in the category of modules for the \mathbb{F}_p -groups or for their infinitesimal subgroups.

Meanwhile, quantized enveloping algebras were discovered by V. G. Drinfeld and Jimbo M. Their representation theory at roots of 1 has subsequently been related to that of affine Kac-Moody Lie algebras by D. Kazhdan and Lusztig [KL1, 2] and [L4], to the former [AJS] has related the representation theory of simple \mathbb{F}_p -groups, and Lusztig's conjectural formula for affine Kac-Moody Lie algebras has been verified by Kashiwara M. and Tanisaki T. [KT]. Altogether Lusztig's conjectural modular irreducible character formula is now proved to hold for large p and in type A , D , and E .

The morphism spaces of modules for simple \mathbb{F}_p -groups are \mathbb{F}_p -linear whereas those for quantized enveloping algebras over cyclotomic fields $\mathbb{Q}(\zeta)$ are $\mathbb{Q}(\zeta)$ -linear, hence one cannot hope to have an equivalence between these categories. Neither is \mathbb{F}_p flat over \mathbb{Z} . In order to overcome the difficulties,

[AJS] works not over \mathbb{F}_p , $\mathbb{Q}(\zeta)$ or \mathbb{Z} , but over various localizations of the completions of the Cartan part of the universal enveloping algebra of the Lie algebra of the \mathbb{F}_p -group and of the quantized enveloping algebra over $\mathbb{Q}(\zeta)$, introduces certain combinatorial categories over these algebras and finally over the symmetric algebra of the root lattice, then applies some standard techniques of finite dimensional algebras.

a° The problem

(a1) Let us first fix the notations.

R an irreducible root system with the set of coroots R^\vee

R^+ a positive system of R

Σ the simple system of R^+

X the weight lattice of R

X^+ the set of dominant weights of X

\geq the standard partial order on X such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$

W the Weyl group of R

$W_a = W \ltimes \mathbb{Z}$ the affine group of W

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$

α_0 the dominant short root of R^+

$h = \langle \rho, \alpha_0^\vee \rangle + 1$ the Coxeter number of R

$(d_\alpha)_{\alpha \in \Sigma} \in \{1, 2, 3\}^\Sigma$ minimal such that $[(d_\alpha \langle \beta, \alpha^\vee \rangle)]_{\alpha, \beta \in \Sigma}$ is symmetric

(a2) Let $k = \mathbb{F}_p$ the prime field of characteristic $p > 0$, and \mathfrak{G}_k the simply connected simple k -group with a maximal torus \mathfrak{T}_k split over \mathbb{Z} and the associated root system R . We will identify X with the weight group $\text{Grp}_k(\mathfrak{T}_k, \mathfrak{GL}_1)$ of \mathfrak{T}_k .

If M is a \mathfrak{T}_k -module, M admits a weight space decomposition $M = \coprod_{\lambda \in X} M_\lambda$ with $M_\lambda = \{m \in M \mid t(m \otimes 1) = m \otimes \lambda(t) \text{ in } M \otimes A \ \forall A \in \mathbf{Alg}_k \text{ and } t \in \mathfrak{T}_k(A)\}$, where \mathbf{Alg}_k denotes the category of commutative k -algebras. One

calls $\lambda \in X$ a weight of M iff $M_\lambda \neq 0$. Set $\text{ch}M = \sum_{\lambda \in X} (\dim M_\lambda) e(\lambda)$, called the character of M , in the group algebra $\mathbb{Z}[X]$ of X with the natural basis $e(\lambda)$, $\lambda \in X$.

There is a bijection, due to C. Chevalley [J], (II.2.4), between X^+ and the set of the isomorphism classes of the simple \mathfrak{G}_k -modules such that

$$(1) \quad \lambda \longmapsto L(\lambda)_k \quad \text{simple of highest weight } \lambda.$$

The fundamental problem in the representation theory of \mathfrak{G}_k has been to find all $\text{ch}L(\lambda)_k$.

(a3) Let $\mathfrak{F}_{\mathfrak{G}} : \mathfrak{G}_k \rightarrow \mathfrak{G}_k$ be the Frobenius endomorphism of \mathfrak{G}_k . Let $X_k = \{\mu \in X^+ \mid \langle \mu, \alpha^\vee \rangle \leq p-1 \ \forall \alpha \in \Sigma\}$. If $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in X_k$ and $\lambda^1 \in X^+$, Steinberg's tensor product theorem says

$$L(\lambda)_k \simeq L(\lambda^0)_k \otimes_k L(\lambda^1)_k^{[1]} \quad \text{in } \mathfrak{G}_k\mathbf{Mod},$$

where $L(\lambda^1)_k^{[1]}$ is the composite of the representation $L(\lambda^1)_k$ with $\mathfrak{F}_{\mathfrak{G}}$. Hence we have only to find all $\text{ch}L(\lambda)_k$, $\lambda \in X_k$.

(a4) Let $\mathfrak{G}_1 = \ker \mathfrak{F}_{\mathfrak{G}}$ the Frobenius kernel of \mathfrak{G}_k . It is an infinitesimal subgroup of \mathfrak{G}_k defined by the Hopf algebra $k[\mathfrak{G}]/\mathfrak{m}_k^p$, where $k[\mathfrak{G}]$ is the Hopf algebra of \mathfrak{G}_k with the augmentation ideal \mathfrak{m}_k . Due to C.W. Curtis [J], (II.3.15),

$$(1) \quad L(\lambda)_k, \lambda \in X_k, \text{ remains simple as } \mathfrak{G}_1\text{-module.}$$

In order to keep track of the weights, however, we will work in the category of $\mathfrak{G}_1\mathfrak{I}_k$ -modules.

In $\mathfrak{G}_1\mathfrak{I}_k\mathbf{Mod}$ the simples are still parametrized by their highest weights, varying though over the entire X . We will denote the simple of highest weight $\mu \in X$ in $\mathfrak{G}_1\mathfrak{I}_k\mathbf{Mod}$ by $L_k(\mu)$. Then

$$(2) \quad L_k(\mu) \simeq L(\mu^0)_k \otimes_k p\mu^1 \quad \text{with} \quad p\mu^1 = (\mu^1)^{[1]}.$$

(a5) Let $\lambda \in X$. If \mathfrak{B}_k is the Borel subgroup of \mathfrak{G}_k whose roots are $-R^+$, regard λ as a \mathfrak{B}_k -module via the projection $\mathfrak{B}_k \rightarrow \mathfrak{I}_k$, and let

$$\begin{aligned} \hat{Z}_k(\lambda) &= \{f \in \mathbf{Sch}_k(\mathfrak{G}_1\mathfrak{I}_k, \mathbb{A}^1) \mid f(A)(xb) = (\lambda(A)(b))^{-1} f(A)(x) \\ &\quad \forall x \in \mathfrak{G}_1\mathfrak{I}_k(A), b \in \mathfrak{B}_1\mathfrak{I}_k(A), A \in \mathbf{Alg}_k\}, \end{aligned}$$

that is just the $\mathfrak{G}_1\mathfrak{T}_k$ -module of global sections of the invertible sheaf on the quotient $\mathfrak{G}_1\mathfrak{T}_k/\mathfrak{B}_1\mathfrak{T}_k$ induced by the $\mathfrak{B}_1\mathfrak{T}_k$ -module λ , where \mathfrak{B}_1 is the Frobenius kernel of \mathfrak{B}_k and \mathbf{Sch}_k denotes the category of k -schemes. The $\mathfrak{G}_1\mathfrak{T}_k$ -module structure is given by $xf = f(x^{-1}?)$. Regarded as a functor $\mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \rightarrow \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ \hat{Z}_k is exact, that makes the representation theory of $\mathfrak{G}_1\mathfrak{T}_k$ more algebraic than that of \mathfrak{G}_k . One has

$$\mathrm{ch}\hat{Z}_k(\lambda) = e(\lambda) \prod_{\alpha \in R^+} \frac{1 - e(-p\alpha)}{1 - e(-\alpha)},$$

hence the composition factor multiplicity $[\hat{Z}_k(\lambda) : L_k(\lambda)] = 1$, and all the other composition factors of $\hat{Z}_k(\lambda)$ have highest weights $< \lambda$. It follows that the determination of $\mathrm{ch}L_k(\lambda)$ is now reduced to counting the decomposition numbers $[\hat{Z}_k(\lambda) : L_k(\mu)]$ for all $\lambda, \mu \in X$.

(a6) Define a partition of X into disjoint subsets, called the blocks of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$, to be the finest partition such that λ and μ belong to the same block if $\mathrm{Ext}_{\mathfrak{G}_1\mathfrak{T}_k}^1(L_k(\lambda), L_k(\mu)) \neq 0$. The linkage principle [J], (II.6.17) says

(1) *each block is contained in a W_a -orbit,*

where we let W_a act on X by $\gamma w \cdot_k \lambda = w(\lambda + \rho) - \rho + p\gamma$, $\gamma \in \mathbb{Z}R$, $w \in W$, and $\lambda \in X$.

If b is a block of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$, denote by $\mathfrak{G}_1\mathfrak{T}_k(b)$ the full subcategory of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ consisting of all modules whose composition factors are of the form $L_k(\lambda)$, $\lambda \in b$. If Ω and Γ are two W_a -orbits in X , one has an exact functor

$$T_{\Omega}^{\Gamma} : \coprod_{b \subseteq \Omega} \mathfrak{G}_1\mathfrak{T}_k(b) \longrightarrow \coprod_{b \subseteq \Gamma} \mathfrak{G}_1\mathfrak{T}_k(b),$$

called the translation functor from Ω to Γ , that is both left and right adjoint to the translation functor T_{Γ}^{Ω} [J], (II.7).

(a7) Let $\mathfrak{A}_k = \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle x + \rho, \alpha_0^{\vee} \rangle < p \ \forall \alpha \in R^+\}$. The W_a -translates of \mathfrak{A}_k are called alcoves. In particular, \mathfrak{A}_k is called the bottom dominant alcove. One has

$$\mathfrak{A}_k \cap X \neq \emptyset \quad \text{iff} \quad 0 \in \mathfrak{A}_k \quad \text{iff} \quad p \geq h.$$

Assume from now on that $0 \in \mathfrak{A}_k$ throughout the rest of the survey.

Let $W_a^+ = \{w \in W_a \mid w \cdot_k 0 \in X^+\}$ and $W_1 = \{w \in W_a \mid w \cdot_k 0 \in X_k\}$. Note that both W_a^+ and W_1 are independent of k .

As $\hat{Z}_k(\lambda)$ is indecomposable, one can write by the linkage principle

$$\text{ch} L_k(\lambda) = \sum_{w \in W_a} a_{\lambda w} \text{ch} \hat{Z}_k(w \cdot_k \lambda), \quad a_{\lambda w} \in \mathbb{Z}.$$

If μ belong to the “upper closure” of the alcove of λ , then the translation principle [J], (II.7.17)(b) yields

$$(1) \quad \text{ch} L_k(\mu) = \sum_{w \in W_a} a_{\lambda w} \text{ch} \hat{Z}_k(w \cdot_k \mu).$$

Also $\hat{Z}_k(\lambda + p\mu) = \hat{Z}_k(\lambda) \otimes_k p\nu \quad \forall \nu \in X$, hence together with (a4)(2)

$$(2) \quad [\hat{Z}_k(\lambda + p\nu) : L_k(\eta + p\nu)] = [\hat{Z}_k(\lambda) : L_k(\eta)].$$

As any weight belongs to the upper closure of an alcove, for $p \geq h$ the problem is now reduced to counting all

$$(3) \quad [\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)], \quad w \in W_a, w' \in W_1.$$

(a8) One says a $\mathfrak{G}_1 \mathfrak{I}_k$ -module admits a \hat{Z}_k -filtration if it has a filtration in $\mathfrak{G}_1 \mathfrak{I}_k \mathbf{Mod}$ with the factors of the form $\hat{Z}_k(\nu)$, $\nu \in X$.

Let $Q_k(\lambda)$ be the projective cover of $L_k(\lambda)$, $\lambda \in X$, in $\mathfrak{G}_1 \mathfrak{I}_k \mathbf{Mod}$. The Brauer-Humphreys reciprocity [J], (II.11.4) says

$$(1) \quad Q_k(\lambda) \text{ admits a } \hat{Z}_k\text{-filtration}$$

and that the multiplicities in the \hat{Z}_k -filtration are given by

$$(2) \quad [Q_k(\lambda) : \hat{Z}_k(w \cdot_k \lambda)] = [\hat{Z}_k(w \cdot_k \lambda) : L_k(\lambda)],$$

where the factors of the filtration must be of the form $\hat{Z}_k(w \cdot_k \lambda)$, $w \in W_a$, by the linkage principle. Hence the problem is further reduced to finding the multiplicities in \hat{Z}_k -filtrations

$$(3) \quad [Q_k(w \cdot_k \lambda) : \hat{Z}_k(w' \cdot_k \lambda)] \quad \forall w \in W_1, w' \in W_a.$$

(a9) Let $\Omega_0 = W_a \cdot_k 0$ and $\lambda \in \Omega_0$. In one case the \hat{Z}_k -filtration of $Q_k(\lambda)$

is well-understood. The Steinberg module $\hat{Z}_k((p-1)\rho) = L_k((p-1)\rho) = L((p-1)\rho)_k$ is a projective indecomposable [J], (II.10.2), hence also

$$\hat{Z}_k((p-1)\rho + p\nu) \simeq \hat{Z}_k((p-1)\rho) \otimes_k p\nu \quad \forall \nu \in X.$$

If λ lies in the top alcove of the box $p\lambda^1 + X_k$, then [J], (II.11.10)

$$(1) \quad Q_k(\lambda) = T_{W_{a \cdot k}((p-1)\rho + p\lambda^1)}^{\Omega_0} \hat{Z}_k((p-1)\rho + p\lambda^1),$$

in a \hat{Z}_k -filtration of which all $\hat{Z}_k(w \cdot_k \lambda^0 + p(\rho - w\rho + \lambda^1))$, $w \in W$, appear exactly once. More generally [J], (II.9.19),

(a10) **Lemma.** *Let $\lambda, \mu \in X$ belonging to the closure of an alcove. Then $T_{W_{a \cdot k}\lambda}^{W_{a \cdot k}\mu} \hat{Z}_k(\lambda)$ has a \hat{Z}_k -filtration with the factors*

$$\hat{Z}_k(w \cdot_k \mu), \quad w \in C_{W_a}(\lambda)/C_{W_a}(\lambda) \cap C_{W_a}(\mu),$$

each appearing exactly once.

(a11) Let Σ_a be the set of reflexions of W_a in a wall of \mathfrak{A}_k , that is independent of k . If $s \in \Sigma_a$, choose $\mu_s \in X \cap \overline{\mathfrak{A}_k}$ with $C_{W_a}(\mu_s) = \{1, s\}$, and set $T_s = T_{\Omega_0}^{W_{a \cdot k}\mu_s}$, $T'_s = T_{W_{a \cdot k}\mu_s}^{\Omega_0}$, and $\Theta_s = T_s \circ T'_s$.

For $\lambda \in \Omega_0$ define a sequence $I = (s_1, \dots, s_r)$ of elements of Σ_a inductively as follows. If λ lies in the top alcove of the box $p\lambda^1 + X_k$, take $I = \emptyset$. Otherwise choose $s_1 \in \Sigma_a$ such that $\lambda < ws_1 \cdot_k 0$ if $\lambda = w \cdot_k 0$, $w \in W_a$, and that $ws_1 \cdot_k 0 \in p\lambda^1 + X_k$. Now set

$$Q_k^I(\lambda) = \Theta_{s_1} \circ \dots \circ \Theta_{s_r} Q_k^\emptyset(\lambda)$$

with $Q_k^\emptyset(\lambda) = T_{W_{a \cdot k}((p-1)\rho + p\lambda^1)}^{\Omega_0} \hat{Z}_k((p-1)\rho + p\lambda^1)$. From (a10) we know the \hat{Z}_k -filtration of $Q_k^I(\lambda)$. On the other hand, if $\hat{\lambda} = w_0 \cdot_k \lambda^0 + p(\lambda^1 + 2\rho)$,

$$(1) \quad Q_k^I(\lambda) = \prod_{\substack{\nu \in \Omega_0 \\ \lambda \uparrow \nu \uparrow \hat{\lambda}}} Q_k(\nu)^{m_k(\lambda, \nu)} \quad \text{with} \quad m_k(\lambda, \lambda) = 1,$$

where \uparrow is a partial order on X such that $\nu \uparrow \nu'$ if $\nu' = s_\beta \cdot_k \nu + pm\beta \geq \nu$ for some $\beta \in R^+$ and $m \in \mathbb{Z}$ [J], (II.11.6).

As the $Q_k(\nu)$ are linearly independent, the $m_k(\lambda, \nu)$ are uniquely determined. Then by induction on $\hat{\lambda} - \lambda$ finding all $m_k(\lambda, \nu)$ will determine the

\hat{Z}_k -filtration of each $Q_k(\nu)$, $\nu \in \Omega_0$.

(a12) The set of $w \in W_a$ with

$$0 \uparrow w \cdot_k 0 \uparrow \widehat{w \cdot_k 0} \uparrow \hat{0} = 2(p-1)\rho$$

is finite and independent of k . Enumerate those w_1, \dots, w_{n_0} such that if $w_i \cdot_k 0 \uparrow w_j \cdot_k 0 \uparrow \widehat{w_j \cdot_k 0} \uparrow \widehat{w_i \cdot_k 0}$, then $j \leq i$. Note that

$$W_1 \subseteq \{w_1, \dots, w_{n_0}\}.$$

For each $w_i \cdot_k 0$, $i \in [1, n_0]$, choose a sequence $I(i)$ as in (a11) and set $Q^{[i]}(k) = Q_k^{I(i)}(w_i \cdot_k 0)$. Then

$$(1) \quad Q^{[i]}(k) = \prod_{j=1}^i Q_k(w_j \cdot_k 0)^{m_k(j,i)} \quad \text{with} \quad m_k(i,i) = 1.$$

Set $Q(k) = \prod_{i=1}^{n_0} Q^{[i]}(k)$ and let

$$\mathcal{E}_{[i],[j]}(k) = \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}(Q^{[i]}(k), Q^{[j]}(k)), \quad \mathcal{E}(k) = \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}(Q(k), Q(k)).$$

Then $\mathcal{E}(k) = \prod_{i,j \in [1, n_0]} \mathcal{E}_{[i],[j]}(k)$. Under the composition each $\mathcal{E}(k)_{[i],[i]}$ and $\mathcal{E}(k)$ form finite dimensional k -algebras.

Let $1 = \sum_{n \in E_k(i)} e_k^n(i)$ be a decomposition into orthogonal primitive idempotents in $\mathcal{E}(k)_{[i],[i]}$, where $E_k(i)$ is an indexing set with $e_k^0(i)$ corresponding to $Q_k(w_i \cdot_k 0)$, i.e., $Q_k(w_i \cdot_k 0) \simeq e_k^0(i)Q^{[i]}(k)$. Then $1 = \sum_{i=1}^{n_0} \sum_{n \in E_k(i)} e_k^n(i)$ is a decomposition into orthogonal primitive idempotents in $\mathcal{E}(k)$. Now

$$(2) \quad \begin{aligned} &e_k^n(i) \text{ is conjugate to } e_k^m(j) \text{ in } \mathcal{E}(k), \text{ i.e., there is some } u \in \mathcal{E}(k)^\times \\ &\text{with } e_k^n(i) = u e_k^m(j) u^{-1}, \text{ iff } \mathcal{E}(k) e_k^n(i) \simeq \mathcal{E}(k) e_k^m(j) \text{ in } \mathcal{E}(k) \mathbf{Mod} \text{ iff} \\ &e_k^n(i) Q(k) \simeq e_k^m(j) Q(k) \text{ in } \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}. \end{aligned}$$

Hence if $n \neq 0$, $e_k^n(i)$ is conjugate to some $e_k^0(j)$ for $j < i$ while $e_k^0(i)$ is not conjugate to any of $e_k^m(j)$, $m \in E_k(j)$ with $j < i$. It follows that

$$(3) \quad m_k(j,i) = \#\{s \in E_k(i) \mid E_k^s(i) \text{ is conjugate to } e_k^0(j) \text{ in } \mathcal{E}(k)\}.$$

(a13) By transferring from $\mathfrak{G}_k \mathbf{Mod}$ to $\mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}$ one has obtained finite

dimensional projectives (in $\mathfrak{G}_k\mathbf{Mod}$ there are no finite dimensional injectives nor projectives), and the translations in W_a have been reflected in a simple manner: for each λ and $\nu \in X$,

$$L_k(\lambda + p\nu) \simeq L_k(\lambda) \otimes_k p\nu, \quad \hat{Z}_k(\lambda + p\nu) \simeq \hat{Z}_k(\lambda) \otimes_k p\nu, \\ \text{and} \quad Q_k(\lambda + p\nu) \simeq Q_k(\lambda) \otimes_k p\nu.$$

In characteristic 0 similar phenomenon occurs with the quantized enveloping algebra.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with v an indeterminate and $U(\mathcal{A})$ Lusztig's \mathcal{A} -form of the Drinfeld-Jimbo quantized enveloping algebra over $\mathbb{Q}(v)$ [L3]. Let $\ell \in \mathbb{N}^+$ prime to the nonzero entries of the Cartan matrix of R , ζ a primitive ℓ -th root of 1 in \mathbb{C} , $\kappa = \mathbb{Q}(\zeta)$, and $U(\kappa) = U(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$. Lusztig has discovered a characteristic 0 analogue of the Frobenius kernel in $U(\kappa)$, that is an $\ell^{|R|}(2\ell)^{|\Sigma|}$ -dimensional subalgebra $\mathfrak{u}(\kappa)$ of $U(\kappa)$ generated by $E_{\pm\alpha}$, K_{α} , $\alpha \in \Sigma$. Let $\mathcal{C}_{U(\kappa)}$ be the category of finite dimensional $U(\kappa)$ -modules with K_{α}^{ℓ} acting by 1 for each $\alpha \in \Sigma$. One has $K_{\alpha}^{2\ell} = 1$ in $U(\kappa)$. Then (cf. [APW1], (9.12); if ℓ is not a prime power, one argues as in [AW]) each $M \in \mathcal{C}_{U(\kappa)}$ admits a weight space decomposition with respect to the Cartan subalgebra $U^0(\kappa) = U^0(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$ with $U^0(\mathcal{A})$ the \mathcal{A} -subalgebra of $U(\mathcal{A})$ generated by $K_{\alpha}^{\pm 1}$ and $[K_{\alpha}] = \prod_{i=1}^m \frac{K_{\alpha} v^{d_{\alpha}(-i+1)} - K_{\alpha}^{-1} v^{-d_{\alpha}(-i+1)}}{v^{d_{\alpha}i} - v^{-d_{\alpha}i}} \otimes 1$, $\alpha \in \Sigma$, $m \in \mathbb{N}$:

$$(1) \quad M = \coprod_{\lambda \in X} M_{\lambda} \quad \text{with} \quad M_{\lambda} = \{m \in M \mid um = \lambda(u)m \ \forall u \in U^0(\kappa)\},$$

$$\text{where } \lambda(K_{\alpha}) = \zeta^{d_{\alpha}\langle \lambda, \alpha^{\vee} \rangle} \text{ and } \lambda([K_{\alpha}]) = \left[\begin{smallmatrix} \langle \lambda, \alpha^{\vee} \rangle \\ m \end{smallmatrix} \right]_{d_{\alpha}} \text{ with } \left[\begin{smallmatrix} r \\ m \end{smallmatrix} \right]_{d_{\alpha}} = \prod_{i=1}^m \frac{v^{d_{\alpha}(r-i+1)} - v^{-d_{\alpha}(r-i+1)}}{v^{d_{\alpha}i} - v^{-d_{\alpha}i}} \otimes 1.$$

The simples of $\mathcal{C}_{U(\kappa)}$ are parametrized by their highest weights in X^+ as in $\mathfrak{G}_k\mathbf{Mod}$. Let $X_{\kappa} = \{\mu \in X^+ \mid \langle \mu, \alpha^{\vee} \rangle \leq \ell - 1 \ \forall \alpha \in \Sigma\}$. If $L(\lambda)_{\kappa}$ denotes the simple of $\mathcal{C}_{U(\kappa)}$ of highest weight $\lambda \in X^+$ and if $\lambda = \lambda^0 + \ell\lambda^1$ with $\lambda^0 \in X_{\kappa}$ and $\lambda^1 \in X$, then Lusztig's tensor product theorem [LMR], (7.4) asserts

$$(2) \quad L(\lambda)_{\kappa} \simeq L(\lambda^0)_{\kappa} \otimes_{\kappa} \bar{L}(\lambda^1)_{\kappa}^{[1]} \quad \text{in } \mathcal{C}_{U(\kappa)},$$

where $\bar{L}(\lambda^1)_{\kappa}^{[1]}$ is the composite of the simple representation $\bar{L}(\lambda^1)_{\kappa}$ of \mathfrak{G}_{κ} , i.e., of the universal enveloping algebra $U(\text{Lie}(\mathfrak{G}_{\kappa}))$ of the Lie algebra $\text{Lie}(\mathfrak{G}_{\kappa})$

of \mathfrak{G}_κ , with Lusztig's lift $U(\kappa) \rightarrow U(\text{Lie}(\mathfrak{G}_\kappa))$ of the Frobenius morphism [L3], (8.16) such that for each $\alpha \in \Sigma$ and $n \in \mathbb{N}$

$$E_{\pm\alpha}^{(n)} \mapsto \begin{cases} \bar{E}_{\pm\alpha}^{(\frac{n}{\ell})} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases} \quad K_\alpha^{\pm 1} \mapsto K_\alpha^{\pm 1}, \quad \begin{bmatrix} K_\alpha \\ n \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} H_\alpha \\ \frac{n}{\ell} \end{bmatrix} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases}$$

where $(\bar{E}_{\pm\beta}, H_\alpha)_{\alpha \in \Sigma, \beta \in R}$ is a basis of $\text{Lie}(\mathfrak{G}_\kappa)$ obtained from a Chevalley basis, and $E_{\pm\alpha}^{(r)} = \frac{E_{\pm\alpha}^r}{[r]_{d_\alpha}!}$ in $U(\kappa)$ with $[r]_{d_\alpha}! = \prod_{i=1}^r \frac{v^{d_\alpha i} - v^{-d_\alpha i}}{v^{d_\alpha} - v^{-d_\alpha}} \otimes 1$ while $\bar{E}_{\pm\alpha}^{(r)} = \frac{\bar{E}_{\pm\alpha}^r}{r!}$ in $U(\text{Lie}(\mathfrak{G}_k))$. By [AW], (1.9)

$$(3) \quad L(\lambda^0)_\kappa \text{ remains simple as } \mathfrak{u}(\kappa)\text{-module.}$$

Again in order to keep track of the weights, we will consider $\tilde{\mathfrak{u}}(\kappa) = U^0(\kappa)\mathfrak{u}(\kappa)$ and the category $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ of all finite dimensional $\tilde{\mathfrak{u}}(\kappa)$ -modules admitting weight space decompositions (1) with K_α^ℓ acting by 1 for each $\alpha \in \Sigma$. The category $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ resembles much the category $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ of finite dimensional $\mathfrak{G}_1\mathfrak{T}_k$ -modules [APW2], (4.7/4.10) (again if ℓ is not a prime power, refer to [AW]). In particular, finding the irreducible characters of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ is reduced for $\ell \geq h$ to the determination of the multiplicity $m_\kappa(j, i)$ of the projective cover $Q_\kappa(w_j \cdot_\kappa 0)$ of $L_\kappa(w_j \cdot_\kappa 0)$ in the projective $Q^{[i]}(\kappa)$:

$$(4) \quad Q^{[i]}(\kappa) = \coprod_{j \leq i} Q_\kappa(w_j \cdot_\kappa 0)^{m_\kappa(j, i)},$$

using the notations of (a12) to define $Q^{[i]}(\kappa)$, where \cdot_κ is the (\cdot_k) -action of W_a on X with p replaced by ℓ . Define $\mathcal{E}_{[i], [j]}(\kappa)$, $\mathcal{E}(\kappa)$, and the idempotents as in (a12) with k replaced by κ . Then

$$(5) \quad m_\kappa(j, i) = \#\{s \in E_\kappa(i) \mid e_\kappa^s(i) \text{ is conjugate to } e_\kappa^0(j) \text{ in } \mathcal{E}(\kappa)\}.$$

(a14) We are not to ask for an equivalence of categories between $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ and $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$, but to expect for p and $\ell \geq h$

$$(1) \quad m_k(i, j) = m_\kappa(i, j) \quad \forall i, j.$$

Indeed, a morphism space in $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ is finite dimensional over \mathbb{F}_p while that in $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ is finite dimensional over $\mathbb{Q}(\zeta)$.

If $p = \ell < h$, however, Andersen and Jantzen have found an example [A1], (7.9) that $\text{ch } L_k(\lambda) \neq \text{ch } L_\kappa(\lambda)$ for some $\lambda \in X_k = X_\kappa$.

b° The theorem

(b1) Retain the notations of (a12/13).

Theorem (cf. [AJS], Corollary 16.8) *There is a \mathbb{Z} -algebra \mathcal{E} of finite type as \mathbb{Z} -module with isomorphisms*

$$\mathcal{E} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}(k) \quad \text{in } k\mathbf{Alg} \quad \text{and} \quad \mathcal{E} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}(\kappa) \quad \text{in } \kappa\mathbf{Alg}.$$

Moreover, \mathcal{E} admits a decomposition $\mathcal{E} = \coprod_{i,j \in [1, n_0]} \mathcal{E}_{[i],[j]}$ such that $\mathcal{E}_{[i],[j]} \mathcal{E}_{[n],[m]} \subseteq \delta_{jn} \mathcal{E}_{[i],[m]}$ for each i, j, m and n , and that the above isomorphisms restrict to isomorphisms

$$\mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}_{[i],[j]}(k) \quad \text{and} \quad \mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}_{[i],[j]}(\kappa),$$

respectively.

(b2) **Remark** (cf. [AJS], Corollary 16.11) *One can realize \mathcal{E} such that $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{d}]$ is free of finite type over $\mathbb{Z}[\frac{1}{d}]$ with $d = (h - 1)!$.*

(b3) For a commutative ring A let us write $\mathcal{E}_A = \mathcal{E} \otimes_{\mathbb{Z}} A$. There is a finite extension field F of \mathbb{Q} that is a splitting field of $\mathcal{E}_{\mathbb{Q}}$ [NT], Theorem 2.3.11. Let \mathfrak{o}_F be the ring of algebraic integers in F and let $1 = \sum_{n \in E_F(i)} e_F^n(i)$, $1 \leq i \leq n_0$, and $1 = \sum_{i=1}^{n_0} \sum_{n \in E_F(i)} e_F^n(i)$ be decompositions into orthogonal primitive idempotents in $(\mathcal{E}_{[i],[i]})_F$ and \mathcal{E}_F , respectively. One can find $N \in \mathbb{N}^+$ such that if $\mathfrak{o} = \mathfrak{o}_F[\frac{1}{N}]$, then (cf. [NT], Lemma 1.13.14)

$$(2) \quad \mathfrak{o} \text{ is of finite type as } \mathbb{Z}[\frac{1}{N}]\text{-module,}$$

$$(3) \quad \mathcal{E}_{\mathfrak{o}} \text{ is } \mathfrak{o}\text{-free of finite type,}$$

$$(4) \quad \text{all } e_F^n(i) \text{ live in } \mathcal{E}_{\mathfrak{o}},$$

i.e., one can write $e_F^n(i) = e^n(i) \otimes 1$ with idempotents $e^n(i)$ in $\mathcal{E}_{\mathfrak{o}}$, and

$$(5) \quad e^n(i) \text{ and } e^m(j) \text{ are conjugate in } \mathcal{E}_F \text{ iff they are so in } \mathcal{E}_{\mathfrak{o}} \quad \forall i, j, n, m.$$

If $\mathfrak{m} \in \text{Max}(\mathfrak{o})$, $\mathfrak{o}_{\mathfrak{m}}$ is a DVR as \mathfrak{o} is a Dedekind domain [AM], (9.5). Put $\mathfrak{o}' = \mathfrak{o}_{\mathfrak{m}}$, $\mathfrak{m}' = \mathfrak{m}\mathfrak{o}'$, and let $\hat{\mathfrak{o}}'$ be the completion of \mathfrak{o}' in the \mathfrak{m}' -adic topology.

Then $\hat{\mathfrak{o}}'$ is a complete DVR with the maximal ideal $\hat{\mathfrak{m}}' = \mathfrak{m}'\hat{\mathfrak{o}}'$ (cf. [B1], (VI.5.3), Proposition 5) and with $\hat{\mathfrak{o}}'/\hat{\mathfrak{m}}' \simeq \mathfrak{o}'/\mathfrak{m}' \simeq \mathfrak{o}/\mathfrak{m}$ [AM], (10.16). In fact, if $\hat{\mathfrak{o}}$ is the completion of \mathfrak{o} in the \mathfrak{m} -adic topology, then $\hat{\mathfrak{o}} \simeq \hat{\mathfrak{o}}'$ [B1], Exercise III.2.27(a). As F is a splitting field of $\mathcal{E}_{\mathbb{Q}}$, the $e^n(i)$ remain primitive in $\mathcal{E}_{\text{Frac}(\hat{\mathfrak{o}}')}$, hence in $\mathcal{E}_{\hat{\mathfrak{o}}'}$. Also $e^n(i)$ and $e^m(j)$ are conjugate in $\mathcal{E}_{\hat{\mathfrak{o}}'}$ iff they are so in $\mathcal{E}_{\mathfrak{o}'}$. Hence (cf. [NT], Theorem 1.14.2(ii))

$$(6) \quad \text{the } e^n(i) \text{ remain primitive in } \mathcal{E}_{\mathfrak{o}/\mathfrak{m}},$$

and (cf. [NT], Theorem 1.14.2(iii))

$$(7) \quad e^n(i) \text{ and } e^m(j) \text{ are conjugate in } \mathcal{E}_{\mathfrak{o}/\mathfrak{m}} \text{ iff they are so in } \mathcal{E}_{\mathfrak{o}}.$$

Rearrange the index sets $E(i)$ of the primitive idempotents in $\mathcal{E}_{\mathfrak{o}}$ so that $e^0(i)$ is not conjugate in $\mathcal{E}_{\mathfrak{o}}$ to any of $e^m(j)$, $m \in E(j)$, $j < i$.

(b4) As the simples of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ are absolutely simple, any indecomposable projective of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ remains indecomposable projective under field extensions. Hence

$$(1) \quad m_{\kappa}(j, i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$$

Also if $p \gg 0$ so that $p \notin \mathfrak{o}^{\times}$, then considering $\mathfrak{m} \in \text{Max}(\mathfrak{o})$ with $p \in \mathfrak{m}$ yields

$$(2) \quad m_k(j, i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$$

Hence for $p \gg 0$

$$(3) \quad m_k(j, i) = m_{\kappa}(j, i).$$

(b5) Let $\mathfrak{u}^{-}(\kappa)$ be the κ -subalgebra of $\mathfrak{u}(\kappa)$ generated by $E_{-\alpha}$, $\alpha \in \Sigma$, and let $\tilde{\mathfrak{u}}^b(\kappa) = \mathfrak{u}^{-}(\kappa)U^0(\kappa)$. Define a category $\mathcal{C}_{\tilde{\mathfrak{u}}^b(\kappa)}$ of finite dimensional $\tilde{\mathfrak{u}}^b(\kappa)$ -modules just like $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$. In analogy to the functor $\hat{Z}_k : \mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \rightarrow \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ one has an induction functor $\tilde{Z}_{\kappa} : \mathcal{C}_{\tilde{\mathfrak{u}}^b(\kappa)} \rightarrow \mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ defined by $\tilde{Z}_{\kappa}(M) = \tilde{\mathfrak{u}}^b(\kappa)\mathbf{Mod}(\tilde{\mathfrak{u}}(\kappa), M)$ [APW2], (1.2). Then

$$(1) \quad \text{ch } \tilde{Z}_{\kappa}(\lambda) = e(\lambda) \prod_{\alpha \in R^+} \frac{1 - e(-\ell\alpha)}{1 - e(-\alpha)} \quad \forall \lambda \in X,$$

and [APW2], (4.10)

$$(2) \quad \text{the Brauer-Humphreys reciprocity carries over to } \mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}.$$

Corollary (cf. [AJS], Corollary 16.23) Assume $\ell \geq h$ and $p \gg 0$ relative to R . Then for each $w, w' \in W_a$ there is $d(w, w') \in \mathbb{N}$ independent of ℓ and p such that

$$[\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)] = d(w, w') = [\tilde{Z}_\kappa(w \cdot_\kappa 0) : L_\kappa(w' \cdot_\kappa 0)].$$

In particular, if $p = \ell$, then

$$ch L_k(w \cdot_k 0) = ch L_\kappa(w \cdot_\kappa 0) \quad \forall w \in W,$$

hence together with the translation principle

$$ch L(\lambda)_k = ch L(\lambda)_\kappa \quad \forall \lambda \in X_k = X_\kappa.$$

(b6) It follows that the irreducible characters of $\mathfrak{G}_k \mathbf{Mod}$ are obtained from that of $\mathcal{C}_{U(\kappa)}$ if $p \gg 0$. Hence from [KL1, 2], [L4] and [KT] Lusztig's conjectural irreducible character formula in $\mathfrak{G}_k \mathbf{Mod}$ holds if $p \gg 0$ and if R is of type A , D or E .

c° Reformulation of categories

(c1) In order to treat much alike categories $\mathfrak{G}_1 \mathfrak{T}_k \mathbf{mod}$ and $\mathcal{C}_{\tilde{u}(\kappa)}$ simultaneously, we will reformulate these categories as follows.

Case 1. Let $k[\mathfrak{G}]$ be the Hopf algebra defining \mathfrak{G}_k and \mathfrak{m}_k the augmentation ideal of $k[\mathfrak{G}]$. Let $\text{Dist}(\mathfrak{G}_k) = \varinjlim_{n \geq 0} \text{Mod}_k(k[\mathfrak{G}]/\mathfrak{m}_k^{n+1}, k)$ the algebra of distributions of \mathfrak{G}_k , that inherits the structure of Hopf algebra from $k[\mathfrak{G}]$.

Any \mathfrak{G}_k -module M is a $k[\mathfrak{G}]$ -comodule, hence a $\text{Dist}(\mathfrak{G}_k)$ -module : if $\Delta_M = id_{k[\mathfrak{G}]} \in \mathfrak{G}_k(k[\mathfrak{G}]) : M \rightarrow M \otimes_k k[\mathfrak{G}]$ is the comodule map, then each $x \in \text{Dist}(\mathfrak{G}_k)$ acts on M by $(M \otimes_k x) \circ \Delta_M$. Conversely, any finite dimensional $\text{Dist}(\mathfrak{G}_k)$ -module carries a structure of \mathfrak{G}_k -module [J], (II.1.20).

The Hopf algebra of \mathfrak{G}_1 is $k[\mathfrak{G}]/\mathfrak{m}_k^p$, hence $\text{Dist}(\mathfrak{G}_1) = (k[\mathfrak{G}]/\mathfrak{m}_k^p)^*$. Then $\mathfrak{G}_1 \mathbf{Mod} = \text{Dist}(\mathfrak{G}_1) \mathbf{Mod}$: if M is a $\text{Dist}(\mathfrak{G}_1)$ -mod, one gets the comodule map by the commutative diagram

$$\begin{array}{ccccc} m & & M & \dashrightarrow & M \otimes_k k[\mathfrak{G}]/\mathfrak{m}_k^p \\ \downarrow & & \downarrow & & \parallel \\ x \mapsto xm & \text{Mod}_k(\text{Dist}(\mathfrak{G}_1), M) & \xrightarrow{\sim} & & M \otimes_k \text{Dist}(\mathfrak{G}_1)^*. \end{array}$$

Let $\mathfrak{g} = \text{Lie}(\mathfrak{G}_k) = \text{Mod}_k(\mathfrak{m}_k/\mathfrak{m}_k^2, k) \leq \text{Dist}(\mathfrak{G}_k)$, and $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ the triangular decomposition with $\mathfrak{h} = \text{Lie}(\mathfrak{T}_k)$. For each $x \in \mathfrak{g}$ one has $x^p \in \mathfrak{g}$ in $\text{Dist}(\mathfrak{G}_k)$ [DG], (II.7.2.3), which we will denote by $x^{[p]}$. In particular [DG], (II.7.2.2), if $x \in \mathfrak{n}^\pm$, then $x^{[p]} = 0$ while if $x \in \mathfrak{h}$, then $x^{[p]} = x$.

If $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} , then

$$(1) \quad x^p - x^{[p]} \in Z(U(\mathfrak{g})),$$

where x^p is the p -th power of x in $U(\mathfrak{g})$. One calls $U^{[p]}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} \mid x \in \mathfrak{g})$ the restricted enveloping algebra of \mathfrak{g} . There is a commutative diagram of k -algebras

$$(2) \quad \begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\text{natural}} & \text{Dist}(\mathfrak{G}_k) \\ \downarrow & & \uparrow \\ U^{[p]}(\mathfrak{g}) & \xrightarrow{\sim} & \text{Dist}(\mathfrak{G}_1). \end{array}$$

Fix a k -basis $(H_\alpha, \bar{E}_\beta \mid \alpha \in \Sigma, \beta \in R)$ of \mathfrak{g} with $H_\alpha = [\bar{E}_\alpha, \bar{E}_{-\alpha}]$ obtained from a Chevalley basis. Let $I = (\bar{E}_\beta^p \mid \beta \in R) \trianglelefteq U(\mathfrak{g})$ and set $\bar{U}(\mathfrak{g}) = U(\mathfrak{g})/I$. The adjoint action of \mathfrak{T}_k on $U(\mathfrak{g})$ stabilizes I , hence $\bar{U}(\mathfrak{g})$ comes equipped with an X -gradation given by the \mathfrak{T}_k -action. As $\bar{E}_\beta^p \in Z(U(\mathfrak{g}))$, $\bar{U}(\mathfrak{g})$ retains a PBW-type basis $(\bar{E}^m H^r \bar{F}^n \mid m, n \in [0, p-1]^{R^+}, r \in \mathbb{N}^\Sigma)$ with

$$\bar{E}^m = \prod_{\beta \in R^+} \bar{E}_\beta^{m_\beta}, H_r = \prod_{\alpha \in \Sigma} H_\alpha^{r_\alpha} \text{ and } \bar{F}^n = \prod_{\beta \in R^+} \bar{E}_{-\beta}^{n_\beta}.$$

The degree of $\bar{E}^m H^r \bar{F}^n$ is $\sum_{\beta \in R^+} (m_\beta - n_\beta)\beta$.

Case 2. Let U_2 be the De Concini-Kac version [DCK], (1.5) of the quantized enveloping algebra over κ , i.e., the κ -algebra with the generators $E_{\pm\alpha}$, $K_\alpha^{\pm 1}$, $\alpha \in \Sigma$, and the same relations as the Drinfeld-Jimbo algebra over $\mathbb{Q}(v)$ with v replaced by ζ . Let U_2^\pm (resp. U_2^0) be the κ -subalgebra of U_2 generated by $E_{\pm\alpha}$ (resp. $K_\alpha^{\pm 1}$), $\alpha \in \Sigma$. For each $w \in W$ let T_w be the endomorphism of U_2 carried over from [LQG]. If $\beta \in R^+$, choose $w \in W$ with $w^{-1}\beta \in \Sigma$, and set $E_\beta = T_w(E_{w^{-1}\beta})$ and $E_{-\beta} = T_w(E_{-w^{-1}\beta})$. In case $\beta \in \Sigma$, the $E_{\pm\beta}$ so defined coincide with the old ones. One can then make U_2 into an X -graded algebra by giving $E_\beta, \beta \in R$ (resp. $K_\alpha, \alpha \in \Sigma$), degree β (resp. 0). By [DCK], Corollary 3.1

$$E_\beta^\ell, K_\alpha^\ell \in Z(U_2) \quad \forall \beta \in R \text{ and } \alpha \in \Sigma.$$

Let $I^\pm = (E_\beta^\ell \mid \beta \in \pm R^+) \trianglelefteq U_2^\pm$ and $I = (I^\pm) \trianglelefteq U_2$. If $f \in \kappa \mathbf{Alg}(U_2, U(\kappa))$ with $E_{\pm\alpha} \mapsto E_{\pm\alpha}$ and $K_\alpha \mapsto K_\alpha$ for each $\alpha \in \Sigma$, then f induces an isomorphism of κ -algebras

$$U_2/(I, K_\alpha^{2\ell} - 1 \mid \alpha \in \Sigma) \simeq \mathfrak{u}(\kappa).$$

Moreover, $I^\pm = \ker(f \mid_{U_2^\pm})$, hence I^\pm are defined independent of the choice of the T_w 's.

Under a suitable choice of the T_w 's and orderings in the products U_2/I retains a PBW-type κ -basis $(E^m K^r F^n \mid m, n \in [0, \ell - 1]^{R^+}, r \in \mathbb{Z}^\Sigma)$ with

$$E^m = \prod_{\beta \in R^+} E_\beta^{m_\beta}, K_r = \prod_{\alpha \in \Sigma} K_\alpha^{r_\alpha} \text{ and } F^n = \prod_{\beta \in R^+} E_{-\beta}^{n_\beta}.$$

(c2) In order to treat the two cases simultaneously, we will denote (κ, ℓ) also by (k, p) and set

$$(U, U^\pm, U^0) = \begin{cases} (\bar{U}(\mathfrak{g}), U(\mathfrak{n}^\pm) + I/I, U(\mathfrak{h}) + I/I) & \text{in Case 1} \\ (U_2/I, U_2^\pm + I/I, U_2^0 + I/I) & \text{in Case 2.} \end{cases}$$

Hence as k -algebras

$$U^0 \simeq \begin{cases} k[H_\alpha \mid \alpha \in \Sigma] & \text{the polynomial algebra in } H_\alpha \text{ in Case 1} \\ k[K_\alpha^{\pm 1} \mid \alpha \in \Sigma] & \text{the Laurent polynomial algebra in } K_\alpha \text{ in Case 2,} \end{cases}$$

and U has

(3) *a structure of k -Hopf algebra (nontrivial in Case 2),*

(4) *a triangular decomposition, i.e.,*

a k -linear bijection $U^- \otimes_k U^0 \otimes_k U^+ \rightarrow U$ under the multiplication,

and

(5) *an X -gradation, indicated by subscripts, such that*

$$U^0 \subseteq U_0, U^+ \subseteq \coprod_{\nu \geq 0} U_\nu, U^- \subseteq \coprod_{\nu \leq 0} U_\nu, \text{ and } (U^+)_0 = k \cdot 1 = (U^-)_0.$$

Define a group homomorphism $\tilde{\cdot} : X \rightarrow \mathbf{Alg}_k(U^0, U^0)^\times$ by

$$\begin{aligned} \tilde{\lambda}(H) &= H + \lambda(H) \quad \forall H \in \mathfrak{h} \quad \text{in Case 1} \\ \tilde{\lambda}(K_\alpha) &= \zeta^{d_\alpha \langle \lambda, \alpha^\vee \rangle} K_\alpha \quad \forall \alpha \in \Sigma \quad \text{in Case 2.} \end{aligned}$$

Then for each $s \in U^0$ and $u \in U_\lambda$ one has $su = u\tilde{\lambda}(s)$.

(c3) Let A be a noetherian domain over U^0 with a structure homomorphism $\pi : U^0 \rightarrow A$ (the assumption that A be a domain is only for convenience in the present survey). We define a category \mathcal{C}_A as follows. An object of \mathcal{C}_A is a $U \otimes_k A$ -module M , which is as A -module of finite type and X -graded. We regard U and A imbedded in $U \otimes_k A$ as $U \otimes 1$ and $1 \otimes A$, respectively, and write $(u \otimes a)m = uma$. We require

$$(1) \quad U_\nu M_\lambda \subseteq M_{\lambda+\nu} \quad \forall \nu \in X$$

and

$$(2) \quad sm = m\pi(\tilde{\lambda}(s)) \quad \forall s \in U^0 \text{ and } m \in M_\lambda.$$

A morphism of \mathcal{C}_A is a morphism of $U \otimes_k A$ -modules that preserves the X -gradings.

The category \mathcal{C}_A is equipped with a duality operation. There is an involutory antiautomorphism τ of U [AJS], (1.6) such that

$$E_\alpha \longmapsto E_{-\alpha} \quad \forall \alpha \in \Sigma \quad \text{and} \quad s \longmapsto s \quad \forall s \in U^0.$$

If $M \in \mathcal{C}_A$, define M^τ to be $\mathbf{Mod}A(M, A)$ with U acting by $(uf)(m) = f(\tau(u)m)$ and with the X -gradation given by

$$(M^\tau)_\lambda = \{f \in M^\tau \mid f(M_\mu) = 0 \quad \forall \mu \neq \lambda\} \simeq \mathbf{Mod}A(M_\lambda, A).$$

If M is A -projective, $(M^\tau)^\tau \simeq M$ in \mathcal{C}_A .

Replacing U by $U^0 U^+$ (resp. U^0) one defines likewise the categories $\mathcal{C}_A^{\geq 0}$ and \mathcal{C}_A^0 .

If $M \in \mathcal{C}_A^0$ is projective in the category of right A -modules $\mathbf{Mod}A$, define the character of M by

$$\text{ch } M = \sum_{\lambda \in X} \text{rk}_A(M_\lambda) e(\lambda) \quad \text{in } \mathbb{Z}[X].$$

(c4) **Case 1.** Take $A = k$ with the structure homomorphism $\pi : U^0 \rightarrow k$ annihilating \mathfrak{h} . Then for each $\lambda \in X$ and $u \in \mathfrak{h}$

$$\tilde{\lambda}(u^p) = \tilde{\lambda}(u)^p = \lambda(u)^p = \lambda(u) = \tilde{\lambda}(u).$$

Hence the U -module structure on $M \in \mathcal{C}_k$ factors through $U^{[p]}(\mathfrak{g})$. Consequently, M comes equipped with a structure of $\text{Dist}(\mathfrak{G}_1)$ -module. Moreover, the X -gradation on M makes M into a \mathfrak{T}_k -module such that

$$t(xm \otimes 1) = (\text{Ad}(t)(x \otimes 1))t(m \otimes 1) \text{ in } M \otimes_k A' \quad \forall t \in \mathfrak{T}_k(A'), A' \in \mathbf{Alg}_k,$$

hence into a $\mathfrak{G}_1\mathfrak{T}_k$ -module. One can thus identify \mathcal{C}_k with $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ the category of finite dimensional $\mathfrak{G}_1\mathfrak{T}_k$ -modules.

Case 2. Take $A = k$ with $\pi : U^0 \rightarrow k$ such that $K_\alpha \mapsto 1 \quad \forall \alpha \in \Sigma$. Then for each $\lambda \in X$ and $\alpha \in \Sigma$

$$\tilde{\lambda}(K_\alpha^p) = \tilde{\lambda}(K_\alpha)^p = \zeta^{pd_\alpha \langle \lambda, \alpha^\vee \rangle} = 1.$$

Hence together with the X -gradation one can identify \mathcal{C}_k with $\mathcal{C}_{\tilde{u}(k)}$.

(c5) The forgetful functor gives an equivalence of categories from \mathcal{C}_A^0 to the category of X -graded A -modules of finite type, hence

(1) \mathcal{C}_A^0 has enough projectives.

Define a functor $\Phi_A : \mathcal{C}_A^0 \rightarrow \mathcal{C}_A$ by setting $\Phi_A(M) = U \otimes_{U^0} M$, $M \in \mathcal{C}_A^0$, with U acting by the left multiplication on U while A acting as given on M . The X -gradation on $\Phi_A(M)$ is defined by $\Phi_A(M)_\lambda = \sum_{\nu \in X} U_\nu \otimes_{U^0} M_{\lambda-\nu}$.

Define likewise a functor $\Phi_A^{\geq 0} : \mathcal{C}_A^0 \rightarrow \mathcal{C}_A^{\geq 0}$ by $\Phi_A^{\geq 0}(M) = U^0 U^+ \otimes_{U^0} M$. Then

(2) Φ_A (resp. $\Phi_A^{\geq 0}$) is exact and

left adjoint to the forgetful functor from \mathcal{C}_A (resp. $\mathcal{C}_A^{\geq 0}$) to \mathcal{C}_A^0 .

Hence from (1)

(3) both \mathcal{C}_A and $\mathcal{C}_A^{\geq 0}$ have enough projectives.

(c6) Define likewise a functor $Z_A : \mathcal{C}_A^{\geq 0} \rightarrow \mathcal{C}_A$ by setting

$$Z_A(M) = U \otimes_{U^0 U^+} M, \quad M \in \mathcal{C}_A^{\geq 0},$$

with the X -gradation on $Z_A(M)$ defined by $Z_A(M)_\lambda = \coprod_{\nu \in X} (U^-)_\nu \otimes_k M_{\lambda-\nu}$, using an A -linear isomorphism $Z_A(M) \simeq U^- \otimes_k M$. Then

(1) Z_A is exact and left adjoint to the forgetful functor $\mathcal{C}_A \rightarrow \mathcal{C}_A^{\geq 0}$

and

$$(2) \quad \Phi_A = Z_A \circ \Phi_A^{\geq 0}.$$

An object of \mathcal{C}_A^0 can be made into an object of $\mathcal{C}_A^{\geq 0}$ through an isomorphism $U^0 U^+ / \coprod_{\nu > 0} (U^0 U^+)_{\nu} \simeq U^0$. In particular, if $\lambda \in X$, define $A^{\lambda} \in \mathcal{C}_A^0$ by

$$(A^{\lambda})_{\nu} = \begin{cases} A & \text{if } \nu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Regarding A^{λ} as an object of $\mathcal{C}_A^{\geq 0}$, set $Z_A(\lambda) = Z_A(A^{\lambda})$. Then

$$(3) \quad \text{ch } Z_A(\lambda) = e(\lambda) \prod_{\beta \in R^+} \frac{1 - e(-p\beta)}{1 - e(-\beta)},$$

that coincides with $\text{ch } \hat{Z}_k(\lambda)$ of §a.

In case $A = F$ is a field

$$(4) \quad Z_F(\lambda) \text{ has a simple head of highest weight } \lambda,$$

which we will denote by $L_F(\lambda)$. All simples of \mathcal{C}_F arise in this way.

(c7) A Z -filtration of $M \in \mathcal{C}_A$ is a chain in \mathcal{C}_A with the successive subquotients isomorphic to some $Z_A(\lambda)$, $\lambda \in X$. By (c6)(3)

- (1) *the multiplicity of $Z_A(\lambda)$ in a Z -filtration is independent of the choice of the Z -filtrations.*

As $\Phi_A = Z_A \circ \Phi_A^{\geq 0}$ and as both Z_A and $\Phi_A^{\geq 0}$ are exact,

- (2) *any $M \in \mathcal{C}_A$ admits an epi $Q \rightarrow M$ in \mathcal{C} with Q projective having a Z -filtration.*

Moreover,

(c8) **Lemma** (cf. [AJS], Lemma 2.16) *If A is local, any direct summand of an object of \mathcal{C}_A with a Z -filtration admits a Z -filtration. In particular, any projective of \mathcal{C}_A has a Z -filtration.*

Proof. One has [AJS], (2.14)

- (1) $\text{Ext}_{\mathcal{C}_A}^1(Z_A(\lambda), Z_A(\mu)) \neq 0$, $\lambda, \mu \in X$, then $\mu > \lambda$.

Let $M = M' \oplus M''$ in \mathcal{C}_A with M having a Z -filtration. If $A = F$ is a field, the standard argument applies: if λ is a maximal weight of M with $r = \dim_F M_\lambda$, then by (1) there is $V \leq M$ with $V \simeq Z_F(\lambda)^{\oplus r}$ such that M/V has a Z -filtration with $[M/V : Z_F(\lambda)] = 0$. If $m \in M'_\lambda \setminus 0$, let $\hat{m} \in \mathcal{C}_F(Z_F(\lambda), M')$ induced by the adjunction from a morphism $F^\lambda \rightarrow M'$ in $\mathcal{C}^{\geq 0}$ such that $1 \mapsto m$. Then $\text{im}(\hat{m}) \leq V$. Denote by \hat{m}' the morphism $Z_F(\lambda) \rightarrow V$ induced from \hat{m} . As $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) \simeq \mathcal{C}_F^{\geq 0}(Z_F(\lambda), \lambda) \simeq F$, $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) = \text{Fid}_{Z_F(\lambda)}$. Hence

$$(2) \quad \hat{m}' \text{ is a split mono with } \text{coker}(\hat{m}') \simeq Z_F(\lambda)^{\oplus r-1}.$$

Then $M'/\text{im}(\hat{m}) \oplus M'' \simeq M/\text{im}(\hat{m}')$ retains a Z_F -filtration. The assertion follows by induction on the length of a Z -filtration on M .

In general, let $\mathfrak{p} \in \text{Spec} A$ with $\kappa(\mathfrak{p})$ the residue field of $A_{\mathfrak{p}}$. As $Z_A(\lambda) \otimes_A \kappa(\mathfrak{p}) \simeq Z_{\kappa(\mathfrak{p})}(\lambda)$ in $\mathcal{C}_{\kappa(\mathfrak{p})}$, it suffices to check by above that

$$(3) \quad \text{if } L \in \mathcal{C}_A \text{ is } A\text{-free with } L_{\kappa(\mathfrak{p})} = L \otimes_A \kappa(\mathfrak{p}) \text{ admitting a } Z\text{-filtration} \\ \text{in } \mathcal{C}_{\kappa(\mathfrak{p})} \text{ for each } \mathfrak{p} \in \text{Spec} A, \text{ then } L \text{ admits a } Z\text{-filtration in } \mathcal{C}_A.$$

Let λ be a maximal weight of L . By (1) again if $s = \dim_{\kappa(\mathfrak{p})}(L_{\kappa(\mathfrak{p})})_\lambda$, there is $L' \leq L_{\kappa(\mathfrak{p})}$ with $L' \simeq Z_{\kappa(\mathfrak{p})}(\lambda)^{\oplus s}$ and such that $L_{\kappa(\mathfrak{p})}$ has a Z -filtration with $[L_{\kappa(\mathfrak{p})}/L' : Z_{\kappa(\mathfrak{p})}(\lambda)] = 0$. As A is local, L_λ remains A -free, say $L_\lambda = Ae_1 \oplus \dots \oplus Ae_s$. If $\hat{e}_1 \in \mathcal{C}_A(Z_A(\lambda), L)$ with $1 \otimes 1 \mapsto e_1$, then as in (2)

$$(4) \quad \hat{e}_1 \otimes_A \kappa(\mathfrak{p}) \text{ is injective and } \text{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p})) \text{ admits a } Z\text{-filtration}.$$

On the other hand, using the duality operator τ of (c3) one has a commutative diagram

$$\begin{array}{ccc} \text{Mod} A(L, A) \otimes_A \kappa(\mathfrak{p}) & \xrightarrow{(\hat{e}_1)^\tau \otimes_A \kappa(\mathfrak{p})} & \text{Mod} A(Z_A(\lambda), A) \otimes_A \kappa(\mathfrak{p}) \\ \wr \downarrow & & \downarrow \wr \\ \text{Mod} \kappa(\mathfrak{p})(L_{\kappa(\mathfrak{p})}, \kappa(\mathfrak{p})) & \xrightarrow{(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau} & \text{Mod} \kappa(\mathfrak{p})(Z_{\kappa(\mathfrak{p})}(\lambda), \kappa(\mathfrak{p})). \end{array}$$

By (4) $(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau$ is surjective, hence $(\hat{e}_1)^\tau \otimes_A A_{\mathfrak{p}}$ is surjective for each $\mathfrak{p} \in \text{Spec} A$ by NAK. Then $(\hat{e}_1)^\tau$ is surjective [AM], (3.9). As $Z_A(\lambda)$ is A -free, the short exact sequence in \mathcal{C}_A

$$0 \rightarrow \ker((\hat{e}_1)^\tau) \rightarrow L^\tau \xrightarrow{(\hat{e}_1)^\tau} Z_A(\lambda)^\tau \rightarrow 0$$

splits in $\mathbf{Mod} A$. Then $\ker((\hat{e}_1)^\tau)$ is A -free as A is local.. Hence $(\ker((\hat{e}_1)^\tau))^\tau$ is A -free in the short exact sequence of \mathcal{C}_A

$$0 \rightarrow Z_A(\lambda) \xrightarrow{\hat{e}_1} L \rightarrow (\ker((\hat{e}_1)^\tau))^\tau \rightarrow 0.$$

By (4) $(\ker((\hat{e}_1)^\tau))^\tau \otimes_A \kappa(\mathfrak{p}) \simeq \text{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))$ has a Z -filtration in $\mathcal{C}_{\kappa(\mathfrak{p})}$ for each $\mathfrak{p} \in \text{Spec} A$. Then by induction on $\text{rk}_A L$ $(\ker((\hat{e}_1)^\tau))^\tau$ admits a Z -filtration, and (3) follows.

The second assertion follows from (c7)(2).

(c9) Define a partition of X into disjoint subsets, called the blocks over A , by taking a finest partition such that λ and μ belong to the same block if either $\mathcal{C}_A(Z_A(\lambda), Z_A(\mu)) \neq 0$ or $\text{Ext}_{\mathcal{C}_A}^1(Z_A(\lambda), Z_A(\mu)) \neq 0$. Let \mathcal{B}_A be the set of blocks over A . Let \mathcal{D}_A be the full subcategory of \mathcal{C}_A consisting of all objects with a Z -filtration. If b is a block over A , let $\mathcal{D}_A(b)$ be the full subcategory of \mathcal{D}_A consisting of all objects such that the subquotients of a Z -filtration are $Z_A(\lambda)$, $\lambda \in b$. Let $\mathcal{C}_A(b)$ be the full subcategory of \mathcal{C}_A consisting of all that are the images of objects of $\mathcal{D}_A(b)$.

(c10) **Theorem** (cf. [AJS], Theorem 6.10) (i) *If b, b' are disjoint blocks over A , then*

$$\text{Ext}_{\mathcal{C}_A}(M, M') = 0 \quad \forall M \in \mathcal{C}_A(b) \text{ and } M' \in \mathcal{C}_A(b').$$

(ii) *Each $M \in \mathcal{C}_A$ admits a block decomposition $M = \coprod_{b \in \mathcal{B}_A} M_b$ with M_b the largest subobject of M belonging to $\mathcal{C}_A(b)$.*

(iii) *For each block b over A the category $\mathcal{C}_A(b)$ is closed under taking homomorphic images, submodules, extensions, and finite direct sums.*

(c11) Relative to the structure homomorphism $\pi : U^0 \rightarrow A$, let

$$R_\pi = \begin{cases} \{\beta \in R \mid \prod_{j=1}^p (\pi(H_\beta) + j) \notin A^\times\} & \text{in Case 1} \\ \{\beta \in R \mid \prod_{j=1}^p (\pi([K_\beta : j])) \notin A^\times\} & \text{in Case 2,} \end{cases}$$

where $[K_\beta : j] = \begin{bmatrix} K_\beta & j \\ 1 & \end{bmatrix} = \frac{K_\beta \zeta^{jd_\beta} - K_\beta^{-1} \zeta^{-jd_\beta}}{\zeta^{d_\beta} - \zeta^{-d_\beta}} (\neq [K_\beta])$ and $d_\beta = d_\alpha$ if $\alpha \in \Sigma$ with $\beta \in W\alpha$. Then R_π forms a root system with the Weyl group

$W_\pi = \langle s_\beta \mid \beta \in R_\pi \rangle$ and a positive system of roots $R_\pi^+ = R_\pi \cap R^+$. Let $W_{\pi,a} = W_\pi \ltimes \mathbb{Z}R_\pi \leq W_a$. It will be convenient to introduce

$$B = \begin{cases} U^0[\frac{1}{\prod_{j=1}^{p-1}(H_\beta+j)} \mid \beta \in R^+] & \text{in Case 1} \\ U^0[\frac{1}{\prod_{j=1}^{p-1}([K_\beta:j])} \mid \beta \in R^+] & \text{in Case 2.} \end{cases}$$

Proposition (cf. [AJS], Proposition 6.13) *Suppose A is a B -algebra. If $b \in \mathcal{B}_A$ and $\lambda \in b$, then $b \subseteq W_{\pi,a} \cdot_k \lambda$.*

(c12) Regard k as a U^0 -algebra via the augmentation. For each $E \in \mathcal{C}_k$ and $M \in \mathcal{C}_A$ one can make $E \otimes_k M$ into an object of \mathcal{C}_A by letting U (resp. A) act via the comultiplication (resp. only on M). The gradation is defined by $(E \otimes_k M)_\lambda = \coprod_{\nu \in X} E_\nu \otimes_k M_{\lambda-\nu}$.

Assume A is a B -algebra. Let W' be a reflexion subgroup of W_a with $W_{\pi,a} \leq W'$. An alcove for W' is a connected component of $X \otimes_{\mathbb{Z}} \mathbb{R}$ with the hyperplanes in W' deleted. Let Ω and Γ be two W' -orbits in X . The closure of an alcove for W' contains exactly one element $\lambda \in \Omega$ and $\mu \in \Gamma$. Then $W(\mu - \lambda)$ is independent of the choice of the alcove. Let ν be the unique dominant weight of $W(\mu - \lambda)$. Choose a simple E of highest weight ν in $\mathfrak{G}_k \mathbf{Mod}$ (resp. $\mathcal{C}_{U(k)}$) in Case 1 (resp. Case 2). Let $\mathcal{C}_A(\Omega) = \coprod_{b \subseteq \Omega} \mathcal{C}_A(b)$ and $\mathcal{C}_A(\Gamma) = \coprod_{b \subseteq \Gamma} \mathcal{C}_A(b)$. If $\text{pr}_\Gamma : \mathcal{C}_A \rightarrow \mathcal{C}_A(\Gamma)$ is the functor such that $\text{pr}_\Gamma M = \coprod_{b \subseteq \Gamma} M_b$, one gets an exact functor

$$T_\Omega^\Gamma = \text{pr}_\Gamma \circ (E \otimes_k ?) : \mathcal{C}_A(\Omega) \longrightarrow \mathcal{C}_A(\Gamma),$$

called the translation functor from Ω to Γ . In case $A = k$ the functor recovers the translation functor in $\mathfrak{G}_k \mathbf{Mod}$ and \mathcal{C}_k . As usual [AJS], (7.6),

$$(1) \quad T_\Omega^\Gamma \text{ is both left and right adjoint to } T_\Gamma^\Omega.$$

Denote the adjunctions by $\text{adj}_1 : \mathcal{C}_A(\Omega)(?, T_\Gamma^\Omega ?') \rightarrow \mathcal{C}_A(\Gamma)(T_\Omega^\Gamma ?, ?')$ and $\text{adj}_2 : \mathcal{C}_A(\Gamma)(?, T_\Omega^\Gamma ?') \rightarrow \mathcal{C}_A(\Omega)(T_\Gamma^\Omega ?, ?')$.

(c13) **Lemma** (cf. [AJS], Lemma 7.5) *Assume A is a B -algebra. Let $\lambda, \mu \in X$ in the closure of an alcove for W' and $\Omega = W' \cdot_k \lambda$, $\Gamma = W' \cdot_k \mu$. Then $T_\Omega^\Gamma Z_A(\lambda)$ has a Z -filtration with factors $Z_A(w \cdot_k \mu)$, $w \in$*

$C_{W'}(\lambda)/C_{W'}(\lambda) \cap C_{W'}(\mu)$, each occuring exactly once.

d° Deformations

(d1) Recall that we are after a characteristic free description of $\mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$. By (c6)(3) and (c8) we may replace $\hat{Z}_k(?)$ of §a by $Z_k(?)$ in \mathcal{C}_k . We will study \mathcal{C}_k by deformations.

Let $\mathfrak{m} \in \text{Spec}(U^0)$ be the annihilator of the trivial 1-dimensional representation:

$$\mathfrak{m} = \begin{cases} (H_\alpha \mid \alpha \in \Sigma) & \text{in Case 1} \\ (K_\alpha - 1 \mid \alpha \in \Sigma) & \text{in Case 2.} \end{cases}$$

Let $\hat{A} = \hat{U}^0$ be the completion of U^0 at \mathfrak{m} , denoted by $A(k)$ in [AJS]. Then \hat{A} is a noetherian complete local domain, flat over U^0 , with maximal ideal $\mathfrak{m}\hat{A}$ and the residue field k . One may regard $\text{Spec}\hat{A}$ as a formal neighbourhood of \mathfrak{m} in $\text{Spec}(U^0)$ (cf. [K], pp. 315-316). Note (cf. [B1], Exercise III.2.27(a)) that \hat{A} is also the completion of B in the $\mathfrak{m}B$ -adic topology.

(d2) **Lemma** (cf. [AJS], Lemma 14.2) *If A is a noetherian complete local domain, the Krull-Schmidt theorem holds in \mathcal{C}_A .*

(d3) Let \mathcal{P}_A be the full subcategory of \mathcal{C}_A consisting of all its projectives.

Theorem (cf. [AJS], Proposition 3.3/Theorem 4.19)) (i) *If $P, Q \in \mathcal{P}_A$, then $\mathcal{C}_A(P, Q)$ is projective of finite type in \mathbf{Mod}_A . If A' is a noetherian domain over A , then in $\mathbf{Mod}_{A'}$*

$$\mathcal{C}_A(P, Q) \otimes_A A' \simeq \mathcal{C}_{A'}(P \otimes_A A', Q \otimes_A A').$$

(ii) *If A is local with the residue field F , then $? \otimes_A F : \mathcal{P}_A \rightarrow \mathcal{P}_F$ gives a bijection between the isomorphism classes.*

(d4) In particular, $Q^{[i]}(k) \in \mathcal{P}_k$ lifts to

$$Q^{[i]}(\hat{A}) = \Theta_{i_1} \circ \dots \circ \Theta_{i_r} \circ T_{\Delta_i}^{\Omega_0} Z_{\hat{A}}(\nu_i)$$

of $\mathcal{P}_{\hat{A}}$, where $\nu_i = (p-1)\rho + p(w_i \cdot_k 0)^1$, $\Omega_0 = W_a \cdot_k 0$, $\Delta_i = W_a \cdot_k \nu_i$, $\Theta_{i_j} = \Theta_{s_{i_j}} = T_{\Gamma_{i_j}}^{\Omega_0} \circ T_{\Omega_0}^{\Gamma_{i_j}}$ with $\Gamma_{i_j} = W_a \cdot_k \mu_{s_{i_j}}$ (cf. (a11/12)). We will see

the projectivity of $Z_{\hat{A}}(\nu_i)$ in (d14). Hence we want now a characteristic free description of $\mathcal{C}_{\hat{A}}(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$.

Let $\hat{A}^\emptyset = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+]$ and $\hat{A}^\beta = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$, $\beta \in R^+$, with $H_\alpha = [K_\alpha : 0]$ in Case 2. Note that \hat{A}^\emptyset and all \hat{A}^β are naturally B -algebras. Put for simplicity $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}}$, $\mathcal{C}_\emptyset = \mathcal{C}_{\hat{A}^\emptyset}$, $\mathcal{C}_\beta = \mathcal{C}_{\hat{A}^\beta}$, and $M^\emptyset = M \otimes_{\hat{A}} \hat{A}^\emptyset$, $M^\beta = M \otimes_{\hat{A}} \hat{A}^\beta$ if $M \in \mathcal{C}_\wedge$. Let also $Z_\wedge(\lambda) = Z_{\hat{A}}(\lambda)$, $Z_\emptyset(\lambda) = Z_{\hat{A}^\emptyset}(\lambda) \simeq Z_\wedge(\lambda)^\emptyset$, and $Z_\beta(\lambda) = Z_{\hat{A}^\beta}(\lambda) \simeq Z_\wedge(\lambda)^\beta$ for each $\lambda \in X$.

(d5) By our standing hypothesis that $p = \text{ch } k \geq h$ in Case 1, we have

Lemma (cf. [AJS], Lemma 9.1) $\hat{A} = \bigcap_{\beta \in R^+} \hat{A}^\emptyset$.

(d6) Let $P, Q \in \mathcal{P}_{\hat{A}}$. As Q is \hat{A} -flat, one may regard $Q \leq Q^\beta \leq Q^\emptyset$ for each $\beta \in R^+$. Then

$$\begin{aligned} & \mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset) \\ & \simeq \mathcal{C}_\wedge(P, Q^\emptyset) \simeq \mathcal{C}_\wedge(P, Q) \otimes_{\hat{A}} \hat{A}^\emptyset \quad \text{as } \hat{A}^\emptyset \text{ is flat over } \hat{A} \text{ (cf. [AJS], Lemma 3.2)} \\ & \geq \mathcal{C}_\wedge(P, Q^\beta) \simeq \mathcal{C}_\wedge(P, Q) \otimes_{\hat{A}} \hat{A}^\beta \quad \text{as } \hat{A}^\beta \text{ is flat over } \hat{A} \\ & \geq \mathcal{C}_\wedge(P, Q). \end{aligned}$$

As $\mathcal{C}_\wedge(P, Q)$ is \hat{A} -flat, one gets from (d5)

$$(1) \quad \mathcal{C}_\wedge(P, Q) = \bigcap_{\beta \in R^+} \mathcal{C}_\beta(P^\beta, Q^\beta) \quad \text{inside } \mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset).$$

(d7) Now \mathcal{C}_\emptyset has a simple structure. If $\text{Frac}(\hat{A})$ is the fractional field of \hat{A} , $\mathcal{C}_{\text{Frac}(\hat{A})}$ is semisimple. To explain that, let us resume the general set-up of \mathcal{C}_A .

Let $w \in W$. Twist $\pi : U^0 \rightarrow A$ by T_w^{-1} to define another U^0 -algebra $A[w]$ with the structure homomorphism $\pi \circ T_w^{-1}$. If $M \in \mathcal{C}_A$, define $M[w] \in \mathcal{C}_{A[w]}$ to be the A -module M with each $u \in U$ acting by $T_w^{-1}(u)$ and the gradation given by $M[w]_\nu = M_{w^{-1}\nu}$. Then the functor $M \mapsto M[w]$ is an equivalence of categories from \mathcal{C}_A to $\mathcal{C}_{A[w]}$. If M is A -projective, then

$$\text{ch}(M[w]) = w(\text{ch } M).$$

Working with the positive system $w(R^+)$ instead of R^+ , define

$$Z_A^w(\lambda) = U \otimes_{U^0 T_w(U^+)} A^\lambda \in \mathcal{C}_A \quad \forall \lambda \in X.$$

Then (cf. [AJS], (4.4)(2)) for each $x \in W$

$$(1) \quad Z_A^x(\lambda)[w] \simeq Z_{A[w]}^{wx}(w\lambda) \quad \text{in } \mathcal{C}_{A[w]},$$

and (cf. [AJS], Lemma 4.10)

$$(2) \quad Z_A(\lambda)^\tau \simeq Z_A^{w_0}(\lambda - 2(p-1)\rho).$$

In particular (cf. [J], (9.2)),

$$(3) \quad Z_k^{w_0}(\lambda) \simeq \hat{Z}_k(\lambda + 2(p-1)\rho) \text{ of } \S a.$$

(d8) Fix $\alpha \in \Sigma$ and put $s = s_\alpha \in \Sigma_\alpha$. Let $U(-\alpha)$ be the subalgebra of U generated by $E_{-\alpha}$, and let $P(\alpha) = U(-\alpha)U^0U^+ \leq U$. Define a full subcategory \mathcal{C}_A^α of $(P(\alpha) \otimes_k A)\mathbf{Mod}$ just like \mathcal{C}_A . Define likewise $Z_A^\alpha(\lambda) = P(\alpha) \otimes_{U^0U^+} A^\lambda$ and $(Z_A^\alpha)^s(\lambda) = P(\alpha) \otimes_{U^0T_s(U^+)} A^\lambda \in \mathcal{C}_A^\alpha$ for each $\lambda \in X$. As the multiplication $U(-\alpha) \otimes_k U^0U^+ \rightarrow P(\alpha)$ is bijective,

(1) $Z_A^\alpha(\lambda)$ (resp. $(Z_A^\alpha)^s(\lambda)$) is A -free of basis

$$v_i = E_{-\alpha}^{(i)} \otimes 1 \text{ (resp. } v'_i = E_\alpha^{(i)} \otimes 1 \text{)},$$

where $E_{-\alpha}^{(i)} = \frac{E_{-\alpha}^i}{i!} \otimes 1$ (resp. $E_\alpha^{(i)} = \frac{E_\alpha^i}{[i]_{d_\alpha}!} \otimes 1$) in Case 1 (resp. Case 2).

One has (cf. [AJS], (5.4))

$$(2) \quad P(\alpha) = U(-\alpha)U^0U(\alpha) \oplus Q(\alpha) \text{ with } Q(\alpha) = \coprod_{\nu \notin \mathbb{Z}\alpha} P(\alpha)_\nu,$$

$$(3) \quad T_s \text{ stabilizes all } P(\alpha), U(-\alpha)U^0U(\alpha) \text{ and } Q(\alpha),$$

and that

$$(4) \quad Q(\alpha) \text{ annihilates both } Z_A^\alpha(\lambda) \text{ and } (Z_A^\alpha)^s(\lambda).$$

Hence one can describe the $P(\alpha)$ -action on both $Z_A^\alpha(\lambda)$ and $(Z_A^\alpha)^s(\lambda)$ explicitly (cf. [AJS], (5.5)). In particular, there is unique

$$(5) \quad \phi_\alpha \in \mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha)) \text{ such that } v_0 \mapsto v'_{p-1}.$$

Then ϕ_α forms an A -basis of $\mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha))$ and one has (cf. [AJS], (5.6))

$$(6) \quad \phi_\alpha(v_i) = \begin{cases} (-1)^i v'_{p-1-i} \binom{\pi(H_\alpha) + \langle \lambda, \alpha^\vee \rangle}{i} & \text{in Case 1} \\ (-1)^i v'_{p-1-i} \pi \left(\begin{bmatrix} K_\alpha i \langle \lambda, \alpha^\vee \rangle \\ i \end{bmatrix} \right) & \text{in Case 2.} \end{cases}$$

It follows that

(7) *if $\alpha \notin R_\pi$, then ϕ_α is bijective.*

If $\alpha \in R_\pi$, let $n_\alpha(\lambda) \in [1, p]$ such that $\pi(H_\alpha) + \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha(\lambda) \cdot 1$ in Case 1 (resp. $\pi(K_\alpha)^2 \zeta^{2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} = \zeta^{2d_\alpha n_\alpha(\lambda)}$ in Case 2). One has (cf. [AJS], (5.9)) that

(8) *if $n_\alpha(\lambda) = p$, then ϕ_α is still bijective.*

(d9) If $w \in W$, from ϕ_α over $A[w^{-1}]$ one gets

$$(1) \quad \phi \in \mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$$

such that the diagram

$$\begin{array}{ccc} Z_A^w(w\lambda) & \xrightarrow{\phi} & Z_A^{ws}(w\lambda - (p-1)w\alpha) \\ \wr \downarrow & & \downarrow \wr \\ Z_{A[w^{-1}]}(\lambda)[w] & & Z_{A[w^{-1}]}^s(\lambda - (p-1)\alpha)[w] \\ \wr \downarrow & & \downarrow \wr \\ (U \otimes_{P(\alpha)} Z_{A[w^{-1}]}^\alpha(\lambda))[w] & \xrightarrow{(U \otimes_{P(\alpha)} \phi_\alpha)[w]} & \{U \otimes_{P(\alpha)} (Z_{A[w^{-1}]}^\alpha)^s(\lambda - (p-1)\alpha)\}[w]. \end{array}$$

commutes. As ϕ sends the standard generator of $Z_A^w(w\lambda)$ to an A -basis element of $Z_A^{ws}(w\lambda - (p-1)w\alpha)_{w\lambda}$,

(2) *ϕ is an A -basis of $\mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$.*

One may compare the construction of ϕ with the intertwining homomorphism

$$H^i(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(s_\alpha \cdot_k \nu)) \longrightarrow H^{i-1}(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(\nu))$$

for $\alpha \in \Sigma$ and $\nu \in X$ with $\langle \nu + \rho, \alpha^\vee \rangle \geq 0$ in $\mathfrak{G}_k \mathbf{Mod} [J]$, (II.5/6).

Choose a reduced expression $w_0 = s_1 s_2 \dots s_N$ of w_0 . If $w_i = s_1 s_2 \dots s_{i-1}$, $1 \leq i \leq N+1$, with $w_1 = 1$, and if $\lambda \langle w_i \rangle = \lambda + (p-1)(w_i \rho - \rho)$, one gets an A -basis ϕ_i of $\mathcal{C}_A(Z_A^{w_i}(\lambda \langle w_i \rangle), Z_A^{w_{i+1}}(\lambda \langle w_{i+1} \rangle))$ like ϕ of (1). One gets from (d8)(7)

(3) *if $R_\pi = \emptyset$, then $Z_A(\lambda) \simeq Z_A^w(\lambda \langle w \rangle) \forall w \in W$,*

i.e., the “Borel-Weil-Bott” theorem holds in \mathcal{C}_A if $R_\pi = \emptyset$.

(d10) Let $\Phi = \phi_N \circ \dots \circ \phi_1 \in \mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$.

Lemma (cf. [AJS], Lemma 5.13) *The morphism Φ is nonzero and forms an A -basis of $\mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$.*

(d11) **Lemma** (cf. [AJS], Lemma 4.9) *If $A = F$ is a field, then*

$$L_F(\lambda) = \text{im } \Phi = \text{soc}_{\mathcal{C}_F} Z_F^{w_0}(\lambda - 2(p-1)\rho).$$

(d12) For each $\beta \in R_\pi$ define $n_\beta \in [1, p]$ as in (d8). One now obtains

Lemma (cf. [AJS], Lemma 6.3) *Assume $A = F$ is a field with the structure homomorphism π .*

- (i) *If $\lambda \in X$ with $n_\beta(\lambda) = p$ for each $\beta \in R_\pi^+$, then $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$ in \mathcal{C}_F .*
- (ii) *If $R_\pi^+ = \emptyset$, then $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$ for each $\lambda \in X$, i.e., \mathcal{C}_F is a semisimple category.*

Proof. As ϕ is bijective, $L_F(\lambda) \simeq Z_F(\lambda)$ for each $\lambda \in X$ by (d11). If $\mu \in X$, then (cf. [AJS], Proposition 4.6)

$$\begin{aligned} \text{Ext}_{\mathcal{C}_F}^1(L_F(\lambda), L_F(\mu)) &\simeq \text{Ext}_{\mathcal{C}_F}^1(L_F(\mu), L_F(\lambda)) \quad \text{using the duality } \tau \\ &\simeq \mathcal{C}_F(\text{rad}_{\mathcal{C}_F} Z_F(\lambda), L_F(\mu)) \quad \text{if } \mu \neq \lambda \\ &= 0. \end{aligned}$$

Hence $L_F(\lambda)$ is both projective and injective in \mathcal{C}_F .

(d13) **Proposition** (cf. [AJS], Corollary 3.5) *Let $M \in \mathcal{C}_A$ with a Z -filtration. Then M is projective in \mathcal{C}_A iff $M \otimes_A (A/\mathfrak{m})$ is projective in $\mathcal{C}_{A/\mathfrak{m}}$ for each maximal ideal \mathfrak{m} of A .*

(d14) We conclude from (d12/13) that for each $\lambda \in X$

- (1) $Z_A((p-1)\rho + p\lambda)$ is projective in \mathcal{C}_A ,
- (2) the block of λ over \hat{A}^\emptyset is a singleton $\{\lambda\}$,

and that

(3) $Z_\emptyset(\lambda)$ is a progenerator of $\mathcal{C}_\emptyset(\{\lambda\})$.

Back to $P, Q \in \mathcal{P}_{\hat{A}}$, one can write $P^\emptyset = \coprod_{\lambda \in X} Z_\emptyset(\lambda)^{p_\lambda}$ and $Q^\emptyset = \coprod_{\lambda \in X} Z_\emptyset(\lambda)^{q_\lambda}$ with $p_\lambda, q_\lambda \in \mathbb{N}$. Then

$$\mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset) \simeq (\hat{A}^\emptyset)^{\sum_{\lambda \in X} p_\lambda q_\lambda}.$$

In particular, if $P^\emptyset = Q^{[i]}(\hat{A}^\emptyset) = Q^{[i]}(\hat{A})^\emptyset$ and $Q^\emptyset = Q^{[j]}(\hat{A}^\emptyset) = Q^{[j]}(\hat{A})^\emptyset$, p_λ (resp. q_λ) are determined independent of k , hence

(4) $\mathcal{C}_\emptyset(Q^{[i]}(\hat{A}^\emptyset), Q^{[j]}(\hat{A}^\emptyset))$ is described independent of k .

(d15) More generally,

Lemma (cf. [AJS], E.4) *Let $\lambda \in X$. For each $M, N \in \mathcal{C}_\emptyset(\{\lambda\})$ one has an isomorphism of \hat{A}^\emptyset -modules*

$$\mathcal{C}_\emptyset(M, N) \longrightarrow \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N)) \quad \text{via} \quad f \longmapsto f \circ ?.$$

Proof. Put $P = Z_\emptyset(\lambda)$ and $M(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M)$, likewise $N(\lambda)$. Consider first the case $M = P^m$ and $N = P^n$ for $m, n \in \mathbb{N}^+$. If $\pi_s : P^m \rightarrow P$ (resp. $i_r : P \rightarrow P^n$) is the projection onto the s -th (resp. injection from the r -th) component, one has a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\ \mathcal{C}_\emptyset(P^m, i_r) \uparrow & & \uparrow \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), \mathcal{C}_\emptyset(P, i_r)) \\ \mathcal{C}_\emptyset(P^m, P) & & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P(\lambda)) \\ \mathcal{C}_\emptyset(\pi_s, P) \uparrow & & \uparrow \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(P, \pi_s), P(\lambda)) \\ \mathcal{C}_\emptyset(P, P) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P(\lambda), P(\lambda)) \\ f & \longmapsto & f \circ ? \end{array}$$

with the bottom horizontal map bijective as $P(\lambda) = \mathcal{C}_\emptyset(P, P) \simeq \hat{A}^\emptyset$. Hence

(1) *the assertion holds with $M = P^m$ and $N = P^n$.*

If N is arbitrary, as P is a generator of $\mathcal{C}_\emptyset(\{\lambda\})$, N admits a finite presentation in $\mathcal{C}_\emptyset(\{\lambda\})$: $P^{n'} \rightarrow P^n \rightarrow N \rightarrow 0$ exact. Then $P^{n'}(\lambda) \rightarrow P^n(\lambda) \rightarrow N(\lambda) \rightarrow 0$ remains exact as $?(\lambda)$ is exact, hence one gets a commutative diagram

$$\begin{array}{ccc}
 \mathcal{C}_\emptyset(P^m, P^{n'}) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^{n'}(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 0 & & 0.
 \end{array}$$

As P^m (resp. $P^m(\lambda) \simeq (\hat{A}^\emptyset)^m$) is a projective of \mathcal{C}_\emptyset (resp. $\mathbf{Mod}_{\hat{A}^\emptyset}$), the left and the right vertical sequences are both exact. By (1) the top and the middle horizontal maps are bijective, hence also the bottom by the 5-lemma, i.e.,

(2) *the assertion holds if $M = P^m$.*

Finally, write $P^{m'} \rightarrow P^m \rightarrow M \rightarrow 0$ exact in \mathcal{C}_\emptyset . One then gets a commutative diagram of exact columns

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(M, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(M(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^{m'}, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^{m'}(\lambda), N(\lambda)).
 \end{array}$$

As the middle and the bottom horizontal maps are bijective by (2), the top horizontal map is bijective by the 5-lemma again, hence the assertion.

(d16) More detailed examination of the ϕ_i and Φ shows that \mathcal{C}_β behaves like \mathfrak{sl}_2 -category. If $\lambda \in X$, $\beta \in R^+$, and if $n \in \mathbb{N}$ minimal with $\langle \lambda + \rho, \beta^\vee \rangle \equiv -n \pmod p$, put $\beta \uparrow \lambda = \lambda + n\beta$.

Theorem (cf. [AJS], Proposition 8.6/Corollary 8.7) *Let $\lambda \in X$ and $\beta \in R^+$.*

- (i) *If $\beta \uparrow \lambda = \lambda$, then $Z_\beta(\lambda)$ is a projective of \mathcal{C}_β .*
- (ii) *Suppose $\beta \uparrow \lambda > \lambda$. Then*

$$\mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)) \simeq \hat{A}^\beta / H_\beta \hat{A}^\beta \quad \text{in } \mathbf{Mod}_{\hat{A}^\beta}.$$

Given a short exact sequence $0 \rightarrow Z_\beta(\beta \uparrow \lambda) \rightarrow Q \rightarrow Z_\beta(\lambda) \rightarrow 0$ in \mathcal{C}_β , Q is projective in \mathcal{C}_β iff the sequence generates $\mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ over \hat{A}^β .

e° Combinatorial categories

(e1) In order to glue together all the \mathfrak{sl}_2 -categories \mathcal{C}_β to recover $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}^\emptyset}$, we introduce a combinatorial category $\mathcal{K}(\Omega)$ for each W_a -orbit Ω . An object of $\mathcal{K}(\Omega)$ is a family $(\mathcal{M}(\lambda))_{\lambda \in \Omega}$ of \hat{A}^\emptyset -modules of finite type, only finitely many nonzero members, together with \hat{A}^β -submodules $\mathcal{M}(\lambda, \beta)$ of finite type for each $\lambda \in \Omega$ and $\beta \in R^+$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $\mathcal{M}(\lambda)$ if $\beta \uparrow \lambda = \lambda$). A morphism of $\mathcal{K}(\Omega)$ is $(\psi_\lambda)_{\lambda \in \Omega} \in \prod_{\lambda \in \Omega} \mathcal{C}_\emptyset(\mathcal{M}(\lambda), \mathcal{M}'(\lambda))$ such that for each $\lambda \in \Omega$ and $\beta \in R^+$

$$\begin{aligned} (\psi_\lambda \oplus \psi_{\beta \uparrow \lambda}) \mathcal{M}(\lambda, \beta) &\subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda > \lambda \\ \psi_\lambda \mathcal{M}(\lambda, \beta) &\subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda = \lambda. \end{aligned}$$

(e2) We define a functor $\mathcal{V}_\Omega : \mathcal{C}_\wedge(\Omega) \rightarrow \mathcal{K}(\Omega)$, that depends on the choice of $e^\beta(\lambda) \in \mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ for $\lambda \in \Omega$ and $\beta \in R^+$ with $\beta \uparrow \lambda > \lambda$. If $M \in \mathcal{C}_\wedge(\Omega)$, set $(\mathcal{V}_\Omega M)(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset)$, and if $\beta \uparrow \lambda = \lambda$, let $(\mathcal{V}_\Omega M)(\lambda, \beta) = \mathcal{C}_\beta(Z_\beta(\lambda), M^\beta)$. If $\beta \uparrow \lambda > \lambda$, represent $e^\beta(\lambda)$ by a short exact sequence in \mathcal{C}_β

$$0 \longrightarrow Z_\beta(\beta \uparrow \lambda) \longrightarrow Q^\beta(\lambda) \longrightarrow Z_\beta(\lambda) \longrightarrow 0.$$

Tensoring with \hat{A}^\emptyset , the sequence splits uniquely to yield an isomorphism $Q^\beta(\lambda)^\emptyset \simeq Z_\emptyset(\beta \uparrow \lambda) \oplus Z_\emptyset(\lambda)$. We set $(\mathcal{V}_\Omega M)(\lambda, \beta)$ to be the image of the composite of the natural maps

$$\begin{aligned} \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) &\longrightarrow \mathcal{C}_\emptyset(Q^\beta(\lambda)^\emptyset, M^\emptyset) \\ &\xrightarrow{\sim} \mathcal{C}_\emptyset(Z_\emptyset(\lambda) \oplus Z_\emptyset(\beta \uparrow \lambda), M^\emptyset) \xrightarrow{\sim} (\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{C}_\Omega M)(\beta \uparrow \lambda). \end{aligned}$$

(e3) Let $\mathcal{FC}_\wedge(\Omega)$ be the full subcategory of $\mathcal{C}_\wedge(\Omega)$ consisting of all \hat{A} -flat objects.

Theorem (cf. [AJS], Proposition 9.4) *Choose all $e^\beta(\lambda)$ as generators of $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$. Then $\mathcal{V}_\Omega : \mathcal{FC}_\wedge(\Omega) \rightarrow \mathcal{K}(\Omega)$ is fully faithful.*

Proof. Let $M, N \in \mathcal{FC}_\wedge(\Omega)$. One must show

$$(\mathcal{V}_\Omega)_{M,N} : \mathcal{C}_\wedge(M, N) \longrightarrow \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$$

is an isomorphism. By (c10) and (d14)(2)

$$\begin{aligned} \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) &= \mathcal{C}_\emptyset\left(\prod_{\lambda \in \Omega} (M^\emptyset)_{\{\lambda\}}, \prod_{\lambda \in \Omega} (N^\emptyset)_{\{\lambda\}}\right) \\ &\simeq \prod_{\lambda} \mathcal{C}_\emptyset((M^\emptyset)_{\{\lambda\}}, \prod_{\mu \in \Omega} (N^\emptyset)_{\{\mu\}}) \simeq \prod_{\lambda} \mathcal{C}_\emptyset((M^\emptyset)_{\{\lambda\}}, (N^\emptyset)_{\{\lambda\}}) \\ &\simeq \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), (M^\emptyset)_{\{\lambda\}}), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), (N^\emptyset)_{\{\lambda\}})) \quad \text{by (d15)} \\ &\simeq \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)). \end{aligned}$$

Hence $(\mathcal{V}_\Omega)_{M,N}$ is injective:

$$\begin{array}{ccc} \mathcal{C}_\wedge(M, N) & \xrightarrow{(\mathcal{V}_\Omega)_{M,N}} & \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N) \\ \wedge | & & \wedge | \\ (3) \quad \mathcal{C}_\wedge(M, N^\emptyset) & \xrightarrow{\quad} & \prod_{\lambda \in \Omega} \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)) \\ \wr | & & \parallel \\ \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) & \xrightarrow{\sim} & \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)) \\ h & \longmapsto & h \circ ?. \end{array}$$

To see the surjectivity, let $\psi \in \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$. By (1) there is $h \in \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset)$ such that for each $\lambda \in \Omega$

$$\psi_\lambda = h \circ ? \quad \text{in } \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)).$$

Let $\beta \in R^+$. For $\lambda \in \Omega$ let $Q^\beta(\lambda)$ be the middle term of the short exact sequence representing $e^\beta(\lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $Z^\beta(\lambda)$ if $\beta \uparrow \lambda = \lambda$). If $\beta \uparrow \lambda > \lambda$, one gets a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) & \dashrightarrow & \mathcal{C}_\beta(Q^\beta(\lambda), N^\beta) \\ \wedge | & & \wedge | \\ \mathcal{C}_\beta(Q^\beta(\lambda), M^\emptyset) & \xrightarrow[u \mapsto h \circ u]{} & \mathcal{C}_\beta(Q^\beta(\lambda), N^\emptyset) \\ \wr | & & | \wr \\ \mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, M^\emptyset) & & \mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, N^\emptyset) \\ \wr | & & | \wr \\ (\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{V}_\Omega M)(\beta \uparrow \lambda) & \xrightarrow[\psi_\lambda \oplus \psi_{\beta \uparrow \lambda}]{} & (\mathcal{V}_\Omega N)(\lambda) \oplus (\mathcal{V}_\Omega N)(\beta \uparrow \lambda), \end{array}$$

hence each $h \circ u$, $u \in \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta)$ factors through N^β . Likewise if $\beta \uparrow \lambda = \lambda$.

On the other hand, one can write by definition $\coprod Z_\beta(\lambda_i) \twoheadrightarrow M^\beta$ with the coproduct running over some $\lambda_i \in \Omega$, hence $\coprod Q^\beta(\lambda_i) \twoheadrightarrow M^\beta$, i.e., there are $u_i \in \mathcal{C}_\beta(Q^\beta(\lambda_i), M^\beta)$ such that $M^\beta = \sum \text{im}(u_i)$. Then

$$h(M^\beta) = h \sum \text{im}(u_i) = \sum \text{im}(h \circ u_i) \subseteq N^\beta.$$

Hence $h(M) = h(\bigcap_{\beta \in R^+} M^\beta) \leq \bigcap_{\beta \in R^+} h(M^\beta) \leq \bigcap_{\beta \in R^+} N^\beta = N$, the last equality following from (d5). Consequently, h arises from $\mathcal{C}_\wedge(M, N)$ with $(\mathcal{V}_\Omega)_{M, N}(h) = \psi$, as desired.

(e4) To get a characteristic free description of $\mathcal{C}_\wedge(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$, it is now enough to find a characteristic free description of

$$\mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A})))$$

with $\Omega_0 = W_a \cdot_k 0$. Define $\mathcal{Z}_{\nu_i}(\hat{A}) \in \mathcal{K}(\Delta_i)$ by setting for each $\mu \in \Delta_i =$

$W_a \cdot k \nu_i$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu) = \begin{cases} \hat{A}^\emptyset & \text{if } \mu = \nu_i \\ 0 & \text{otherwise,} \end{cases}$$

and for each $\beta \in R^+$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu, \beta) = \begin{cases} \hat{A}^\beta & \text{if } \mu = \nu_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(1) \quad \mathcal{V}_{\Delta_i}(Z_\wedge(\nu_i)) \simeq \mathcal{Z}_{\nu_i}(\hat{A}).$$

(e5) We want next to construct a translation functor $\mathcal{T} : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Gamma)$ for W_a -orbits Ω and Γ such that $\mathcal{V}_\Gamma \circ T_\Omega^\Gamma \simeq \mathcal{T} \circ \mathcal{V}_\Omega$. We will consider only the case that for each $\lambda \in \Omega$ there is a unique $\mu \in \Gamma$ that lies in the closure of the facet of λ , which we will denote by λ_Γ .

Put $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$. For each $\lambda \in \Omega$ choose an isomorphism $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$. Let $\beta \in R^+$ with $\beta \uparrow \lambda > \lambda$. Define

$$t[f_\lambda, f_{\beta \uparrow \lambda}] : \text{Ext}_{\mathcal{C}_\wedge}^1(Z_\wedge(\lambda), Z_\wedge(\beta \uparrow \lambda)) \longrightarrow \text{Ext}_{\mathcal{C}_\wedge}^1(Z_\wedge(\lambda_\Gamma), Z_\wedge((\beta \uparrow \lambda)_\Gamma))$$

by sending each short exact sequence $0 \rightarrow Z_\wedge(\beta \uparrow \lambda) \rightarrow Q \rightarrow Z_\wedge(\lambda) \rightarrow 0$ to the bottom horizontal exact sequence of the commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & TZ_\wedge(\beta \uparrow \lambda) & \longrightarrow & TQ & \longrightarrow & TZ_\wedge(\lambda) \longrightarrow 0 \\ & & \uparrow f_{\beta \uparrow \lambda} & & \parallel & & \uparrow f_\lambda \\ 0 & \longrightarrow & Z_\wedge((\beta \uparrow \lambda)_\Gamma) & \longrightarrow & TQ & \longrightarrow & Z_\wedge(\lambda_\Gamma) \longrightarrow 0, \end{array}$$

where the top horizontal sequence is the one obtained by hitting T on the first exact sequence.

Assume first $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$, so that $(\beta \uparrow \lambda)_\Gamma = \beta \uparrow \lambda_\Gamma$. Let $W_\beta = \langle s_\beta \rangle \rtimes \mathbb{Z}\beta \leq W_a$ and let $T_0 = T_{W_{\beta \cdot k\lambda}}^{W_{\beta \cdot k\lambda_\Gamma}}$, $T'_0 = T_{W_{\beta \cdot k\lambda_\Gamma}}^{W_{\beta \cdot k\lambda}}$. As $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$,

$$T'_0 Z_\beta(\lambda_\Gamma) \simeq Z_\beta(\lambda) \quad \text{and} \quad T'_0 Z_\beta(\beta \uparrow \lambda_\Gamma) \simeq Z_\beta(\beta \uparrow \lambda).$$

Let $f'_\lambda = \text{adj}_1^{-1}(f_\lambda^{-1}) \in \mathcal{C}_\beta(Z_\beta(\lambda), T'_0 Z_\beta(\lambda_\Gamma))^\times$ using the adjunction adj_1 with respect to T_0 and T'_0 , and likewise $f'_{\beta \uparrow \lambda}$. Define

$$t[f'_\lambda, f'_{\beta \uparrow \lambda}] : \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma)) \longrightarrow \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$$

just like $t[f_\lambda, f_{\beta \uparrow \lambda}]$ replacing T by T'_0 . Then (cf. [AJS], (10.6)(1))

$$(1) \quad t[f_\lambda, f_{\beta \uparrow \lambda}] \circ t[f'_\lambda, f'_{\beta \uparrow \lambda}] = \text{id}_{\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma))}.$$

Suppose we have chosen an \hat{A}^β -generator $e^\beta(\lambda)$ (resp. $e^\beta(\lambda_\Gamma)$) of

$$\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)) \quad (\text{resp. } \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma))).$$

Then one can write with some a_λ^β and $b_\lambda^\beta \in \hat{A}^\beta$

$$t[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = a_\lambda^\beta e^\beta(\lambda_\Gamma) \quad \text{and} \quad t[f'_\lambda, f'_{\beta \uparrow \lambda}]e^\beta(\lambda_\Gamma) = b_\lambda^\beta e^\beta(\lambda).$$

By (1) and (d16)

$$(2) \quad a_\lambda^\beta b_\lambda^\beta \in 1 + H_\beta \hat{A}^\beta \quad \text{in } \hat{A}^\beta / H_\beta \hat{A}^\beta.$$

Assume next $\beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma$. Define an isomorphism

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}] \in \mathbf{Mod}_{\hat{A}^\beta}(\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)), \hat{A}^\beta H_\beta^{-1} / \hat{A}^\beta)^\times$$

as follows (cf. [AJS], Proposition 8.14). Let $e \in \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ represented by a short exact sequence

$$0 \longrightarrow Z_\beta(\beta \uparrow \lambda) \xrightarrow{i} Q \xrightarrow{j} Z_\beta(\lambda) \longrightarrow 0.$$

As $eH_\beta = 0$ in $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ by (d16), there is a unique $j' \in \mathcal{C}_\beta(Z_\beta(\lambda), Q)$ such that $j \circ j' = H_\beta \text{id}_{Z_\beta(\lambda)}$ (cf. [B2], (X.119) Proposition 4/ (X.120) Corollary 3(ii)): one has a commutative diagram of short exact sequences with the top sequence representing eH_β

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_\beta(\beta \uparrow \lambda) & \longrightarrow & Z_\beta(\beta \uparrow \lambda) \oplus Z_\beta(\lambda) & \longrightarrow & Z_\beta(\lambda) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow H_\beta \\ 0 & \longrightarrow & Z_\beta(\beta \uparrow \lambda) & \xrightarrow{i} & Q & \xrightarrow{j} & Z_\beta(\lambda) \longrightarrow 0. \end{array}$$

As $TZ_\beta(\lambda) \simeq Z_\beta(\lambda_\Gamma) \simeq TZ_\beta(\beta \uparrow \lambda)$, Te splits by (d16) to yield $i' \in \mathcal{C}_\beta(TQ, TZ_\beta(\beta \uparrow \lambda))$ such that $i' \circ Ti = \text{id}_{TZ_\beta(\beta \uparrow \lambda)}$. Then one can write

$$f_{\beta \uparrow \lambda}^{-1} \circ i' \circ Tj' \circ f_\lambda = a \text{id}_{Z_\beta(\lambda_\Gamma)} \quad \text{for some } a \in \hat{A}^\beta.$$

One can check a is independent of the choice of the representative of e (cf. [AJS], (8.13)). Set

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e = a.$$

Then one can write with some $\bar{a}_\lambda^\beta \in \hat{A}^\beta H_\beta^{-1}$ and $\bar{b}_\lambda^\beta \in \hat{A}^\beta H_\beta$

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = \bar{a}_\lambda^\beta + \hat{A}^\beta \quad \text{and} \quad \theta[f_\lambda, f_{\beta \uparrow \lambda}]^{-1}\left(\frac{1}{H_\beta} + \hat{A}^\beta\right) = \bar{b}_\lambda^\beta \frac{1}{H_\beta} e^\beta(\lambda),$$

in which case $\bar{a}_\lambda^\beta \bar{b}_\lambda^\beta \in 1 + \hat{A}^\beta H_\beta$ in $\hat{A}^\beta / \hat{A}^\beta H_\beta$.

Define now $\mathcal{T} : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Gamma)$ and $\mathcal{T}' : \mathcal{K}(\Gamma) \rightarrow \mathcal{K}(\Omega)$ as follows. If $\mathcal{M} \in \mathcal{K}(\Omega)$, set

$$(\mathcal{T}\mathcal{M})(\mu) = \coprod_{\substack{\lambda \in \Omega \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda) \quad \forall \mu \in \Gamma,$$

and for each $\beta \in R^+$ set $(\mathcal{T}\mathcal{M})(\mu, \beta) =$

$$\begin{cases} \coprod_{\substack{\lambda \in \Omega, \beta \uparrow \lambda = \lambda \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda, \beta) \oplus \coprod_{\substack{\lambda \in \Omega, \beta \uparrow \lambda > \lambda \\ \lambda_\Gamma = \mu = (\beta \uparrow \lambda)_\Gamma}} ((\bar{b}_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta \uparrow \mu = \mu \\ \coprod_{\substack{\lambda \in \Omega \\ \lambda_\Gamma = \mu}} ((\bar{b}_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta \uparrow \mu > \mu, \end{cases}$$

where $\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda) \cap \mathcal{M}(\lambda, \beta)$. If $\mathcal{N} \in \mathcal{K}(\Gamma)$, set

$$(\mathcal{T}'\mathcal{N})(\lambda) = \mathcal{N}(\lambda_\Gamma) \quad \forall \lambda \in \Omega,$$

and for each $\beta \in R^+$ set $(\mathcal{T}'\mathcal{N})(\lambda, \beta) =$

$$\begin{cases} \mathcal{N}(\lambda_\Gamma, \beta) & \text{if } \beta \uparrow \lambda = \lambda \\ (a_\lambda^\beta, 1)\mathcal{N}(\lambda_\Gamma, \beta) + \mathcal{N}(\lambda_\Gamma)_\beta & \text{if } \beta \uparrow \lambda_\Gamma > \lambda_\Gamma \\ \mathcal{N}(\lambda_\Gamma, \beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta) & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma + p\beta \\ \{(x + \bar{a}_\lambda^\beta y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\} & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma \text{ and } \beta \uparrow \lambda > \lambda. \end{cases}$$

Although \mathcal{T} and \mathcal{T}' depend on the choices of a_λ^β , b_λ^β , and \bar{a}_λ^β in their classes modulo \hat{A}^β , and \bar{b}_λ^β modulo $\hat{A}^\beta H_\beta^2$, the restriction of \mathcal{T} (resp. \mathcal{T}') to the image of \mathcal{V}_Ω (resp. \mathcal{V}_Γ) is independent of those choices (cf. [AJS], Remark 10.10).

(e6) **Proposition** (cf. [AJS], Proposition 10.11) *One has natural isomorphisms*

$$\mathcal{V}_\Gamma \circ \mathcal{T} \simeq \mathcal{T} \circ \mathcal{V}_\Omega \quad \text{and} \quad \mathcal{V}_\Omega \circ \mathcal{T}' = \mathcal{T}' \circ \mathcal{V}_\Gamma.$$

(e7) Let \mathcal{H} be the set of reflexion hyperplanes for the \cdot_k -action of W_a on

$X \otimes_{\mathbb{Z}} \mathbb{R}$. If $H = \{\nu \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \nu + \rho, \gamma^\vee \rangle = mp\}$, $\gamma \in R^+$ and $m \in \mathbb{Z}$, then we set $\gamma = \alpha(H)$. Also we will write for each $\nu \in X \otimes_{\mathbb{Z}} \mathbb{R}$

$$\nu \geq H \quad \text{iff} \quad \langle \nu + \rho, \gamma^\vee \rangle \geq mp.$$

If $\beta \in R^+$, let $\mathcal{H}(\beta) = \{H \in \mathcal{H} \mid s_\beta(\alpha(H)) < 0\}$. If $\lambda, \mu \in X$ with μ lying in the closure of the facet of λ , set in the fractional field $\text{Frac}(\hat{A})$ of \hat{A}

$$C^\beta(\lambda, \mu) = \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda > H}} h_{-\alpha(H)} \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda < H}} \frac{1}{h_{\alpha(H)}},$$

where $h_\alpha = d_\alpha H_\alpha$ in Case 1 (resp. $\log K_\alpha = \sum_{j \geq 1} \frac{(-1)^{j+1}}{j} (K_\alpha - 1)^j$ in Case 2) for each $\alpha \in R$. If $H_\beta^\mu \in \mathcal{H}$ with $\alpha(H_\beta^\mu) = \beta$ and $\mu \in H_\beta^\mu$, then

$$C^\beta(\lambda, \mu) \in \begin{cases} (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu > \mu \\ \frac{1}{h_\beta} (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda < H_\beta^\mu \\ h_\beta (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda > H_\beta^\mu. \end{cases}$$

(e8) Let $s \in \Sigma_a$, $\mu_s \in X \cap \mathfrak{A}_k$ with $C_{W_a}(\mu_s) = \{1, s\}$, and $\Gamma_s = W_a \cdot_k \mu_s$. Let $\mathcal{G} = \{\Gamma_s, \Delta_i \mid s \in \Sigma_a, i \in [1, n_0]\}$. We can now state a highlight of [AJS], difficult

Theorem of good choice (cf. [AJS], Theorem 13.4) *For $\lambda \in \Omega_0$ let $\lambda_\Gamma \in \Gamma$ in the closure of the alcove of λ , $\Gamma \in \mathcal{G}$. One can simultaneously choose \hat{A}^β -generators $e^\beta(\mu)$ of $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\mu), Z_\beta(\beta \uparrow \mu))$ for each $\mu \in \Omega_0 \cup (\cup_{\Gamma \in \mathcal{G}} \Gamma)$ and $\beta \in R^+$ with $\beta \uparrow \mu > \mu$, and $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$ for each $\lambda \in \Omega_0$ and $\Gamma \in \mathcal{G}$ such that for each $\lambda \in \Omega_0$, $\beta \in R^+$ and $\Gamma \in \mathcal{G}$*

$$t[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma)e^\beta(\lambda_\Gamma) \quad \text{if } \beta \uparrow \lambda_\Gamma > \lambda_\Gamma,$$

and

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma) + \hat{A}^\beta \quad \text{if } \beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma.$$

(e9) With the good choices of the $e^\beta(\lambda)$'s redefine functors \mathcal{V}_{Ω_0} , \mathcal{V}_{Δ_i} , $i \in [1, n_0]$, \mathcal{V}_{Γ_s} , $s \in \Sigma_a$, so that the combinatorial functors \mathcal{T}_s , \mathcal{T}'_s , and \mathcal{T}'_i corresponding to $T_{\Omega_0}^{\Gamma_s}$, $T_{\Gamma_s}^{\Omega_0}$, and $T_{\Delta_i}^{\Omega_0}$, respectively, involve only the constants

$C^\beta(\lambda, \lambda_\Gamma)$, $\lambda \in \Omega_0$, $\Gamma \in \mathcal{G}$. If we set $\mathcal{Q}^{[i]}(\hat{A}) = \mathcal{T}_{i_1} \mathcal{T}'_{i_1} \dots \mathcal{T}_{i_r} \mathcal{T}'_{i_r} \mathcal{T}'_i \mathcal{Z}_{\nu_i}(\hat{A})$ with $\mathcal{T}_{i_j} = \mathcal{T}_{s_{i_j}}$ and $\mathcal{T}'_{i_j} = \mathcal{T}'_{s_{i_j}}$ (cf. (d4)), then

$$\mathcal{Q}^{[i]}(\hat{A}) \simeq \mathcal{V}_{\Omega_0}(\mathcal{Q}^{[i]}(\hat{A})) \quad \text{in } \mathcal{K}(\Omega_0).$$

(e10) Let $S = S(\mathbb{Z}R)$ the symmetric algebra of $\mathbb{Z}R$. Put $S_k = S \otimes_{\mathbb{Z}} k$. Recall the h_α from (e7). If \hat{S}_k is the completion of S_k with respect to the maximal ideal generated by all $\alpha \in R$, one has a k -algebra isomorphism

$$(1) \quad \hat{S}_k \longrightarrow \hat{A} \quad \text{via} \quad \alpha \longmapsto h_\alpha \quad \forall \alpha \in R.$$

Through the isomorphism one can regard $C^\beta(\lambda, \lambda_\Gamma)$ living in $\text{Frac}(S)$ for each $\lambda \in \Omega_0$ and $\Gamma \in \mathcal{G}$. Hence one can define combinatorial categories $\mathcal{K}(\Omega_0, S)$, $\mathcal{K}(\Gamma_s, S)$, $\mathcal{K}(\Delta_i, S)$, combinatorial translation functors \mathcal{T}_s , \mathcal{T}'_s , \mathcal{T}'_i between them, and $\mathcal{Z}_{\nu_i}(S)$, $\mathcal{Q}^{[i]}(S)$ by copying the definitions of $\mathcal{K}(\Omega_0)$, etc., with \hat{A} replaced by S and with $S^\emptyset = S[\frac{1}{\alpha} \mid \alpha \in R^+]$ and $S^\beta = S[\frac{1}{\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$. Note that $h_\alpha \in H_\alpha \hat{A}^\times$.

More generally, let $A \in \mathbf{Alg}_S$ with $\alpha \neq 0$ in A for any $\alpha \in R$. For a W_a -orbit Ω define $\mathcal{K}(\Omega, A)$ likewise. If $A' \in \mathbf{Alg}_A$ with $\alpha \neq 0$ in A' for all $\alpha \in R$, define a functor of extension of scalars $\mathcal{K}(\Omega, A) \rightarrow \mathcal{K}(\Omega, A')$, written $\mathcal{M} \mapsto \mathcal{M}_{A'}$, by

$$\mathcal{M}_{A'}(\lambda) = \mathcal{M}(\lambda) \otimes_{A^\emptyset} A'^\emptyset \simeq \mathcal{M}(\lambda) \otimes_A A' \quad \forall \lambda \in \Omega$$

and for each $\beta \in R^+$ by setting $\mathcal{M}_{A'}(\lambda, \beta)$ equal to the image of

$$\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A'^\beta \simeq \mathcal{M}(\lambda, \beta) \otimes_A A'$$

in $\mathcal{M}_{A'}(\lambda) \oplus \mathcal{M}_{A'}(\beta \uparrow \lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $\mathcal{M}_{A'}(\lambda)$ if $\beta \uparrow \lambda = \lambda$). The translation functors \mathcal{T}_s , \mathcal{T}'_s , \mathcal{T}'_i commute with functors of extension of scalars. In particular,

$$(2) \quad \mathcal{Q}^{[i]}(S)_{S_k} \simeq \mathcal{Q}^{[i]}(S_k) \quad \text{and} \quad \mathcal{Q}^{[i]}(S)_{\hat{A}} \simeq \mathcal{Q}^{[i]}(\hat{A}).$$

Note that

$$(3) \quad \mathcal{K}(\Omega, A) \text{ is independent of } k.$$

For let $\mu \in \Omega \cap \bar{\mathfrak{A}}_k$ and $W_\Omega = C_{W_a}(\mu)$. Define a category $\mathcal{K}(W_a/W_\Omega, A)$ just like $\mathcal{K}(\Omega, A)$. An object of $\mathcal{K}(W_a/W_\Omega, A)$ is a family of A^\emptyset -modules $(\mathcal{M}(wW_\Omega))_{wW_\Omega \in W_a/W_\Omega}$, almost all members 0, together with A^β -submodules

$\mathcal{M}(wW_\Omega, \beta)$, $wW_\Omega \in W_a/W_\Omega$ and $\beta \in R^+$, of $\mathcal{M}(wW_\Omega) \oplus \mathcal{M}((\beta \uparrow w)W_\Omega)$ if $(\beta \uparrow w)W_\Omega \neq wW_\Omega$ (resp. $\mathcal{M}(wW_\Omega)$ if $(\beta \uparrow w)W_\Omega = wW_\Omega$), where $\beta \uparrow w \in W_a$ such that $(\beta \uparrow w) \cdot_k 0 = \beta \uparrow (w \cdot_k 0)$. Then one has an isomorphism

$$\mathcal{K}(\Omega, A) \longrightarrow \mathcal{K}(W_a/W_\Omega, A) \quad \text{via} \quad \mathcal{M} \longmapsto \mathcal{M}'$$

with $\mathcal{M}'(wW_\omega) = \mathcal{M}(w \cdot_k \mu)$ and $\mathcal{M}'(wW_\Omega, \beta) = \mathcal{M}(w \cdot_k \mu, \beta)$ for each $w \in W_a$ and $\beta \in R^+$.

(e11) **Lemma** (cf. [AJS], Lemma 14.8) *If A' is flat over A , then for each $\mathcal{M}, \mathcal{N} \in \mathcal{K}(\Omega, A)$*

$$\mathcal{K}(\Omega, A)(\mathcal{M}, \mathcal{N}) \otimes_A A' \simeq \mathcal{K}(\Omega, A')(\mathcal{M}_{A'}, \mathcal{N}_{A'}).$$

(e12) **Theorem** (cf. [AJS], Lemma 14.9) *Assume $p \gg 0$ in Case 1. Then for each $i, j \in [1, n_0]$,*

$$\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)).$$

Proof. We first rewrite the left hand side as $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_{\mathbb{Z}} k$. For each $\lambda \in \Omega_0$ and $\beta \in R^+$ let

$$\mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 = \begin{cases} (\mathcal{Q}^{[i]}(S)(\lambda) \oplus \mathcal{Q}^{[i]}(S)(\beta \uparrow \lambda)) / \mathcal{Q}^{[i]}(\lambda, \beta) & \text{if } \beta \uparrow \lambda > \lambda \\ \mathcal{Q}^{[i]}(S)(\lambda) / \mathcal{Q}^{[j]}(S)(\lambda, \beta) & \text{if } \beta \uparrow \lambda = \lambda. \end{cases}$$

One has (cf. [AJS], Lemma 14.15(b)/(14.16)) for each $\lambda \in \Omega_0$ and $\beta \in R^+$

$$(1) \quad \mathcal{Q}^{[i]}(S)(\lambda) \text{ is } S^0\text{-free of finite rank,}$$

$$(2) \quad \mathcal{Q}^{[i]}(S)(\lambda, \beta) \text{ is } S^\beta\text{-free of finite rank,}$$

and

$$(3) \quad \mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 \text{ has no } p\text{-torsion.}$$

Consider a natural map

$$\begin{aligned} \psi : \coprod_{\lambda \in \Omega_0} \text{Mod}_{S^0}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) &\longrightarrow \\ \coprod_{\lambda \in \Omega_0} \coprod_{\beta \in R^+} \text{Mod}_{S^\beta}(\mathcal{Q}^{[i]}(S)(\lambda, \beta), \mathcal{Q}^{[j]}(S)(\lambda, \beta)^0). \end{aligned}$$

Then $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) = \ker \psi$. By (3) the codomain of ψ has no p -torsion, hence $\text{im } \psi$ has no p -torsion. Then $\text{Tor}_1^{\mathbb{Z}}(\text{im } \psi, k) = 0$. On the other hand, for each $\lambda \in \Omega_0$

$$\begin{aligned} & \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)) \\ & \simeq \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset, \mathcal{Q}^{[j]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset) \quad \text{by (e10)(2)} \\ & \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{S^\emptyset} S_k^\emptyset \quad \text{by (1)} \\ & \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k. \end{aligned}$$

Hence if ψ_k is the analogue of ψ over S_k , one gets a commutative diagram of short exact sequences

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (\ker \psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda \in \Omega_0} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k & & \\ & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & \ker(\psi_k) & \longrightarrow & \coprod_{\lambda \in \Omega_0} \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)) & \longrightarrow & (\text{im } \psi) \otimes_{\mathbb{Z}} k \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & \longrightarrow \text{im } (\psi_k) \longrightarrow 0. \end{array}$$

Consequently,

$$(5) \quad \mathcal{K}(\Omega, S)(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k \text{ injects into} \\ \mathcal{K}(\Omega, S_k)(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)).$$

Suppose $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$. Then from (2) one gets as in (4) a commutative diagram of exact sequences

$$(6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (\text{im } \psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda, \beta} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda, \beta), \mathcal{Q}^{[j]}(S)(\lambda, \beta)^0) \otimes_{\mathbb{Z}} k & & \\ & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & \text{im } (\psi_k) & \longrightarrow & \coprod_{\lambda, \beta} \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda, \beta), \mathcal{Q}^{[j]}(S_k)(\lambda, \beta)^0). & & \end{array}$$

As the left vertical arrow is surjective by (4), $(\text{im } \psi) \otimes_{\mathbb{Z}} k \simeq \text{im } (\psi_k)$. Then the 5-lemma applied to (4) yields $(\ker \psi) \otimes_{\mathbb{Z}} k \simeq \ker(\psi_k)$, i.e., the bijectivity in (5).

Finally, $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$ automatically in Case 2. In Case 1 $\text{coker } \psi$ is of finite type in \mathbf{Mod}_S , hence (cf. [M], Theorem 6.5) $|\text{Ass}(\text{coker } \psi)| < \infty$. Also (cf. [M], Theorem 6.1) $\text{coker } \psi$ has a p -torsion iff $p \in \bigcup_{\mathfrak{p} \in \text{Ass}(\text{coker } \psi)} \mathfrak{p}$. As each \mathfrak{p} contains a unique prime of \mathbb{Z} , $\text{coker } \psi$ has no p -torsion for $p \gg 0$, in which case $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$.

(e13) **Remark** Theorem e.12 holds, in fact, for $p \geq h$ (cf. [AJS], Theorem 16.7). Its proof, however, requires introduction of \mathbb{Z} -graded combinatorial categories $\tilde{\mathcal{K}}(\Omega_0, S)$ and $\tilde{\mathcal{K}}(\Omega_0, S_k)$ (cf. [AJS], Lemma 16.6), that are also relevant to the question of the Koszulity of $U^{[p]}(\mathfrak{g})$ and $\mathfrak{u}(k)$ in [AJS], §§17/18.

(e14) Regarding \mathbb{Z} as S -algebra via $\alpha \mapsto 0$ for each $\alpha \in \Sigma$, let $\mathcal{E}_{[i],[j]}(\mathbb{Z}) = \mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S \mathbb{Z}$.

Corollary *Let $i, j \in [1, n_0]$.*

(i) $\mathcal{E}_{[i],[j]}(\mathbb{Z})$ *is independent of k .*

(ii) *One has a k -linear isomorphism $\mathcal{E}_{[i],[j]}(\mathbb{Z}) \otimes_{\mathbb{Z}} k \simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$.*

Proof. (i) follows from (e10)(3). The left hand side of (ii) is isomorphic to

$$\begin{aligned}
& \mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \\
& \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)) \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \quad \text{by (e12/13)} \\
& \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k)_{\hat{A}}, \mathcal{Q}^{[j]}(S_k)_{\hat{A}}) \otimes_{\hat{A}} k \quad \text{by (e11) as } \hat{A} \simeq \hat{S}_k \text{ is flat over } S_k \\
& \simeq \mathcal{K}(\Omega_0, \hat{A})(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e10)(2)} \\
& = \mathcal{K}(\Omega_0)(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \\
& \simeq \mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A}))) \otimes_{\hat{A}} k \\
& \simeq \mathcal{C}_{\wedge}(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e3)} \\
& \simeq \mathcal{C}_k(Q^{[i]}(\hat{A}) \otimes_{\hat{A}} k, Q^{[j]}(\hat{A}) \otimes_{\hat{A}} k) \quad \text{by (d3) as } Q^{[i]}(\hat{A}) \text{ is projective} \\
& \simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k)).
\end{aligned}$$

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